Multivector and Dyadic Identities

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This collection of identities is an upgraded Appendix A of the book *Differential Forms in Electromagnetics*, (Wiley and IEEE Press, 2004) by this author. Notation follows that of the book. Where not otherwise stated, n is the dimension of the vector space and p, q are numbers in the range $0...n$.

Notation

$$\text{vectors}\ \ \mathbf{a},\mathbf{b},\mathbf{c},...\in\mathbb{E}_1,\qquad \text{dual vectors}\ \ \boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},...\in\mathbb{F}_1$$

$$\text{bivectors}\ \ \mathbf{A},\mathbf{B},\mathbf{C},...\in\mathbb{E}_2,\qquad \text{dual bivectors}\ \ \boldsymbol{\Phi},\boldsymbol{\Psi},\boldsymbol{\Gamma},...\in\mathbb{F}_2$$

$$p\text{−vectors}\ \ \mathbf{a}^p,\mathbf{b}^p,...\in\mathbb{E}_p,\qquad \text{dual } p\text{−vectors}\ \ \boldsymbol{\alpha}^p,\boldsymbol{\beta}^p,...\in\mathbb{F}_p$$

$$p\text{−index } J=i_1i_2\cdots i_p,\quad N=123\cdots n,\quad \mathbf{a}_N=\mathbf{a}_1\wedge\mathbf{a}_2\cdots\wedge\mathbf{a}_n\in\mathbb{E}_n$$

$$\text{dimension of p}-\text{vectors}\quad C_p^n=\frac{n!}{p!(n-p)!}=C_{n-p}^n$$

$$\text{vector basis}\quad \mathbf{e}_1,\mathbf{e}_2,\ ...,\mathbf{e}_n\in\mathbb{E}_1,\qquad \text{reciprocal dual}-\text{vector basis}\ \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,...,\boldsymbol{\varepsilon}_n\in\mathbb{F}_1$$

$$\text{dyadics}\quad \overline{\overline{\mathsf{A}}},\overline{\overline{\mathsf{B}}},...\in\mathbb{E}_1\mathbb{F}_1,\qquad \overline{\overline{\mathsf{A}}}^T,\overline{\overline{\mathsf{B}}}^T,...\in\mathbb{F}_1\mathbb{E}_1$$

$$\text{metric dyadics}\quad \overline{\overline{\mathsf{A}}},\overline{\overline{\mathsf{B}}},...\in\mathbb{E}_1\mathbb{E}_1,\qquad \overline{\overline{\mathsf{\Gamma}}},\overline{\overline{\mathsf{\Pi}}},...\in\mathbb{F}_1\mathbb{F}_1$$

Multivectors

Multiplication

$$\mathbf{a}^p|\boldsymbol{\alpha}^p=\boldsymbol{\alpha}^p|\mathbf{a}^p\in\mathbb{E}_0$$

$$\mathbf{a}^p\wedge\mathbf{b}^q=(-1)^{pq}\mathbf{b}^q\wedge\mathbf{a}^p\in\mathbb{E}_{p+q},\quad =0\text{ for }p+q>n$$

$$p>q,\qquad \mathbf{a}^p\lfloor\boldsymbol{\alpha}^q=(-1)^{q(p-q)}\boldsymbol{\alpha}^q\rfloor\mathbf{a}^p\in\mathbb{E}_{p-q}$$

$$p=q+r,\qquad (\mathbf{a}^p\lfloor\boldsymbol{\alpha}^q)|\boldsymbol{\beta}^r=\mathbf{a}^p|(\boldsymbol{\alpha}^q\wedge\boldsymbol{\beta}^r)$$

$$p>q+r,\qquad (\mathbf{a}^p\lfloor\boldsymbol{\alpha}^q)\lfloor\boldsymbol{\beta}^r=\mathbf{a}^p\lfloor(\boldsymbol{\alpha}^q\wedge\boldsymbol{\beta}^r)$$

Reciprocal bases

$$\mathbf{e}_i|\boldsymbol{\varepsilon}_j = \boldsymbol{\varepsilon}_j|\mathbf{e}_i = \delta_{ij}$$

$$\mathbf{e}_{K(i)} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_{i-1} \wedge \mathbf{e}_{i+1} \wedge \cdots \wedge \mathbf{e}_n \in \mathbb{E}_{n-1}$$

$$\mathbf{e}_{K(ij)} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_{i-1} \wedge \mathbf{e}_{i+1} \wedge \cdots \wedge \mathbf{e}_{j-1} \wedge \mathbf{e}_{j+1} \wedge \cdots \wedge \mathbf{e}_n \in \mathbb{E}_{n-2}$$

$$\mathbf{e}_N = \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n = (-1)^{i-1}\mathbf{e}_i \wedge \mathbf{e}_{K(i)} = (-1)^{n-i}\mathbf{e}_{K(i)} \wedge \mathbf{e}_i$$

$$\mathbf{e}_N\lfloor\boldsymbol{\varepsilon}_i = (-1)^{i-1}\mathbf{e}_{K(i)}, \qquad \mathbf{e}_N\lfloor\boldsymbol{\varepsilon}_{K(i)} = (-1)^{n-i}\mathbf{e}_i$$

$$\mathbf{e}_N\lfloor\boldsymbol{\varepsilon}_{ij} = (-1)^{i+j-1}\mathbf{e}_{K(ij)}, \qquad \mathbf{e}_N\lfloor\boldsymbol{\varepsilon}_{K(ij)} = (-1)^{i+j-1}\mathbf{e}_{ij}$$

$$(\boldsymbol{\alpha}\rfloor\mathbf{e}_N) \wedge \mathbf{a} = (\boldsymbol{\alpha}|\mathbf{a})\mathbf{e}_N,$$

$$\mathbf{e}_N\lfloor(\mathbf{a}^p\rfloor\boldsymbol{\varepsilon}_N) = \mathbf{a}^p, \qquad \mathbf{e}_N\lfloor(\boldsymbol{\varepsilon}_N\lfloor\mathbf{a}^p) = (-1)^{p(n-p)}\mathbf{a}^p$$

$$n = 4, \qquad \mathbf{e}_N\lfloor(\boldsymbol{\varepsilon}_{123}\lfloor\mathbf{a}^p) = \mathbf{a}^p \wedge \mathbf{e}_4, \quad p < 3$$

$$n = 4, \qquad \mathbf{e}_N\lfloor(\mathbf{a}\rfloor\boldsymbol{\Phi}) = -\mathbf{a} \wedge (\mathbf{e}_N\lfloor\boldsymbol{\Phi}), \quad \boldsymbol{\Phi} \in \mathbb{F}_2$$

$$n = 4, \qquad \mathbf{e}_N\lfloor(\boldsymbol{\varepsilon}_i \wedge \boldsymbol{\alpha}) = (-1)^{i-1}\mathbf{e}_{K(i)}\lfloor\boldsymbol{\alpha}$$

Bivector products

$$(\mathbf{a} \wedge \mathbf{b})|(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (\mathbf{a}|\boldsymbol{\alpha})(\mathbf{b}|\boldsymbol{\beta}) - (\mathbf{a}|\boldsymbol{\beta})(\mathbf{b}|\boldsymbol{\alpha})$$

$$(\mathbf{a} \wedge \mathbf{b})|(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = \mathbf{a}|\{\mathbf{b}\rfloor(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})\} = \{(\mathbf{a} \wedge \mathbf{b})\lfloor\boldsymbol{\alpha}\}|\boldsymbol{\beta}$$

$$(\mathbf{a} \wedge \mathbf{b})\lfloor\boldsymbol{\alpha} = \mathbf{b}(\mathbf{a}|\boldsymbol{\alpha}) - \mathbf{a}(\mathbf{b}|\boldsymbol{\alpha}) = (\mathbf{ba} - \mathbf{ab})|\boldsymbol{\alpha} = -\boldsymbol{\alpha}\rfloor(\mathbf{a} \wedge \mathbf{b})$$

$$\boldsymbol{\alpha} \wedge (\mathbf{a}\rfloor(\boldsymbol{\beta} \wedge \boldsymbol{\gamma})) = -(\mathbf{a}\rfloor(\boldsymbol{\beta} \wedge \boldsymbol{\gamma})) \wedge \boldsymbol{\alpha} = \mathbf{a}\rfloor(\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma}) - (\mathbf{a}|\boldsymbol{\alpha})(\boldsymbol{\beta} \wedge \boldsymbol{\gamma})$$

Bac cab rules

$$\boldsymbol{\alpha}\rfloor(\mathbf{b} \wedge \mathbf{c}) = \mathbf{b}(\boldsymbol{\alpha}|\mathbf{c}) - \mathbf{c}(\boldsymbol{\alpha}|\mathbf{b}) = (\mathbf{c} \wedge \mathbf{b})\lfloor\boldsymbol{\alpha}$$

$$\boldsymbol{\alpha}\rfloor(\mathbf{b} \wedge \mathbf{C}) = \mathbf{b} \wedge (\boldsymbol{\alpha}\rfloor\mathbf{C}) + \mathbf{C}(\boldsymbol{\alpha}|\mathbf{b}) = (\mathbf{C} \wedge \mathbf{b})\lfloor\boldsymbol{\alpha}, \quad \mathbf{C} \in \mathbb{E}_2$$

$$\boldsymbol{\alpha}\rfloor(\mathbf{B} \wedge \mathbf{C}) = \mathbf{B} \wedge (\boldsymbol{\alpha}\rfloor\mathbf{C}) + \mathbf{C} \wedge (\boldsymbol{\alpha}\rfloor\mathbf{B}) = -(\mathbf{B} \wedge \mathbf{C})\lfloor\boldsymbol{\alpha}, \quad \mathbf{B}, \mathbf{C} \in \mathbb{E}_2$$

$$\mathbf{A}\rfloor(\boldsymbol{\beta} \wedge \boldsymbol{\Gamma}) = \boldsymbol{\beta}(\mathbf{A}|\boldsymbol{\Gamma}) + \boldsymbol{\Gamma}\lfloor(\mathbf{A}\lfloor\boldsymbol{\beta}), \quad \mathbf{A} \in \mathbb{E}_2, \boldsymbol{\Gamma} \in \mathbb{F}_2$$

$$\boldsymbol{\alpha}\rfloor(\mathbf{b}^p \wedge \mathbf{c}^q) = \mathbf{b}^p \wedge (\boldsymbol{\alpha}\rfloor\mathbf{c}^q) + (-1)^{pq}\mathbf{c}^q \wedge (\boldsymbol{\alpha}\rfloor\mathbf{b}^p)$$

$$(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})\rfloor(\mathbf{B} \wedge \mathbf{C}) = \mathbf{B}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})|\mathbf{C} + \mathbf{C}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})|\mathbf{B} + (\boldsymbol{\beta}\rfloor\mathbf{B}) \wedge (\boldsymbol{\alpha}\rfloor\mathbf{C}) + (\boldsymbol{\beta}\rfloor\mathbf{C}) \wedge (\boldsymbol{\alpha}\rfloor\mathbf{B})$$

$$(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})\rfloor(\mathbf{b}^p \wedge \mathbf{c}^q)$$

$$= (\mathbf{b}^p\lfloor(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})) \wedge \mathbf{c}^q + \mathbf{b}^p \wedge ((\boldsymbol{\alpha} \wedge \boldsymbol{\beta})\rfloor\mathbf{c}^q) + (\boldsymbol{\alpha}\rfloor\mathbf{b}^p) \wedge (\mathbf{c}^q\lfloor\boldsymbol{\beta}) - (\boldsymbol{\beta}\rfloor\mathbf{b}^p) \wedge (\mathbf{c}^q\lfloor\boldsymbol{\alpha})$$

Trivector products

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = ((\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\lfloor\boldsymbol{\alpha})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = \boldsymbol{\alpha}|((\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\lfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}))$$
$$= \boldsymbol{\alpha}|(\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b}))|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})$$
$$= (\mathbf{a}|\boldsymbol{\alpha})(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) - (\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}))|(\boldsymbol{\alpha}\rfloor(\mathbf{b}\wedge\mathbf{c}))$$
$$(\mathbf{a}\wedge\mathbf{C})|(\boldsymbol{\alpha}\wedge\boldsymbol{\Gamma}) = (\mathbf{a}|\boldsymbol{\alpha})(\mathbf{C}|\boldsymbol{\Gamma}) - (\mathbf{a}\rfloor\boldsymbol{\Gamma})|(\boldsymbol{\alpha}\rfloor\mathbf{C})$$
$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\lfloor\boldsymbol{\alpha} = \{(\mathbf{a}\wedge\mathbf{b})\mathbf{c} + (\mathbf{b}\wedge\mathbf{c})\mathbf{a} + (\mathbf{c}\wedge\mathbf{a})\mathbf{b}\}|\boldsymbol{\alpha}$$
$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\lfloor\boldsymbol{\Gamma} = \{\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b})\}|\boldsymbol{\Gamma}$$
$$n=4, \quad \boldsymbol{\alpha}\wedge(\boldsymbol{\gamma}\lfloor\mathbf{A}) = (\boldsymbol{\alpha}\rfloor\mathbf{A})\rfloor\boldsymbol{\gamma} + (\boldsymbol{\alpha}\wedge\boldsymbol{\gamma})\lfloor\mathbf{A}, \quad \boldsymbol{\gamma}\in\mathbb{F}_3,\ \mathbf{A}\in\mathbb{E}_2$$
$$n=4, \quad \boldsymbol{\alpha}\wedge(\boldsymbol{\varepsilon}_N\lfloor\mathbf{k}) = \boldsymbol{\varepsilon}_N\lfloor(\boldsymbol{\alpha}\rfloor\mathbf{k}), \quad \mathbf{k}\in\mathbb{E}_3$$

Quadrivector products

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta}) = ((\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})\lfloor\boldsymbol{\alpha})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta})$$
$$= \boldsymbol{\alpha}|(\mathbf{a}(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}) - \mathbf{b}(\mathbf{c}\wedge\mathbf{d}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{d}\wedge\mathbf{a}\wedge\mathbf{b}) - \mathbf{d}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}))|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta})$$
$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})\lfloor\boldsymbol{\alpha} = \mathbf{d}\wedge((\mathbf{b}\wedge\mathbf{c})\mathbf{a} + (\mathbf{c}\wedge\mathbf{a})\mathbf{b} + (\mathbf{a}\wedge\mathbf{b})\mathbf{c})|\boldsymbol{\alpha} - (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})(\mathbf{d}|\boldsymbol{\alpha})$$
$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})\lfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta}) =$$
$$= (((\mathbf{a}(\mathbf{b}\wedge\mathbf{c} + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b}))\wedge\mathbf{d} - \mathbf{d}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}))|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta})$$
$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta}) = (\boldsymbol{\gamma}\wedge\boldsymbol{\delta})|((\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}))$$
$$= (\boldsymbol{\gamma}\wedge\boldsymbol{\delta})|((\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d}) - (\mathbf{a}\wedge\mathbf{c})(\mathbf{b}\wedge\mathbf{d}) + (\mathbf{a}\wedge\mathbf{d})(\mathbf{b}\wedge\mathbf{c})$$
$$+(\mathbf{b}\wedge\mathbf{c})(\mathbf{a}\wedge\mathbf{d}) - (\mathbf{b}\wedge\mathbf{d})(\mathbf{a}\wedge\mathbf{c}) + (\mathbf{c}\wedge\mathbf{d})(\mathbf{a}\wedge\mathbf{b}))|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})$$
$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})\lfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}) = (((\mathbf{b}\wedge\mathbf{c})\mathbf{a} + (\mathbf{c}\wedge\mathbf{a})\mathbf{b} + (\mathbf{a}\wedge\mathbf{b})\mathbf{c})\wedge\mathbf{d})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})$$
$$-(\mathbf{d}\wedge(\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b})))|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})$$

Special cases for $n=3$

$$\mathbf{a}(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}) - \mathbf{b}(\mathbf{c}\wedge\mathbf{d}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{d}\wedge\mathbf{a}\wedge\mathbf{b}) - \mathbf{d}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}) = 0$$
$$(\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b}))\wedge\mathbf{d} = \mathbf{d}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})$$
$$((\mathbf{a}\wedge\mathbf{b})\mathbf{c} + (\mathbf{c}\wedge\mathbf{a})\mathbf{b} + (\mathbf{b}\wedge\mathbf{c})\mathbf{a})\wedge\mathbf{d} = \mathbf{d}\wedge(\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b}))$$
$$((\mathbf{a}\wedge\mathbf{b})\rfloor\boldsymbol{\kappa})\wedge((\mathbf{b}\wedge\mathbf{c})\rfloor\boldsymbol{\kappa}) = (\mathbf{b}\rfloor\boldsymbol{\kappa})((\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})|\boldsymbol{\kappa}), \quad \boldsymbol{\kappa}\in\mathbb{F}_3$$

Quintivector products

$$\boldsymbol{\alpha}\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}\wedge\mathbf{e}) = \boldsymbol{\alpha}|(\mathbf{a}(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}\wedge\mathbf{e}) - \mathbf{b}(\mathbf{a}\wedge\mathbf{c}\wedge\mathbf{d}\wedge\mathbf{e})$$
$$+\mathbf{c}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{d}\wedge\mathbf{e}) - \mathbf{d}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{e}) + \mathbf{e}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}))$$
$$= (\boldsymbol{\alpha}\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}))\wedge(\mathbf{d}\wedge\mathbf{e}) + (\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\wedge(\boldsymbol{\alpha}\rfloor(\mathbf{d}\wedge\mathbf{e}))$$

Special cases for $n = 4$

$$\mathbf{a}(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}\wedge\mathbf{e}) - \mathbf{b}(\mathbf{a}\wedge\mathbf{c}\wedge\mathbf{d}\wedge\mathbf{e}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{d}\wedge\mathbf{e})$$
$$-\mathbf{d}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{e}) + \mathbf{e}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}) = 0$$
$$(((\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b}))\wedge\mathbf{d}) - \mathbf{d}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}))\wedge\mathbf{e} + \mathbf{e}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}) = 0$$
$$(\boldsymbol{\alpha}\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}))\wedge(\mathbf{d}\wedge\mathbf{e}) = -(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})\wedge(\boldsymbol{\alpha}\rfloor(\mathbf{d}\wedge\mathbf{e}))$$

p-vector rules

$$(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdots\wedge\mathbf{a}_p)\lfloor\boldsymbol{\alpha} = (-1)^{p-1}\boldsymbol{\alpha}\rfloor(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdots\wedge\mathbf{a}_p) = \boldsymbol{\alpha}|\sum_{i=1}^{p}(-1)^{i-1}\mathbf{a}_i\mathbf{a}_{K_p(i)}$$
$$= \boldsymbol{\alpha}|(\mathbf{a}_1\mathbf{a}_{K_p(1)} - \mathbf{a}_2\mathbf{a}_{K_p(2)} + \cdots + (-1)^{p-1}\mathbf{a}_p\mathbf{a}_{K_p(p)}), \qquad 2 \le p \le n$$
$$\mathbf{a}_{K_p(i)} = \mathbf{a}_1\wedge\cdots\wedge\mathbf{a}_{i-1}\wedge\mathbf{a}_{i+1}\wedge\cdots\wedge\mathbf{a}_p$$
$$(\mathbf{a}^p\rfloor\boldsymbol{\varepsilon}_N)|(\boldsymbol{\alpha}^p\rfloor\mathbf{e}_N) = \mathbf{a}^p|\boldsymbol{\alpha}^p$$

Dyadics

Basic rules

$$\mathbf{a}\boldsymbol{\alpha}\in\mathbb{E}_1\mathbb{F}_1, \quad \mathbf{a}\mathbf{b}\in\mathbb{E}_1\mathbb{E}_1, \quad (\mathbf{a}\boldsymbol{\alpha})^T = \boldsymbol{\alpha}\mathbf{a}\in\mathbb{F}_1\mathbb{E}_1, \quad \boldsymbol{\alpha}\boldsymbol{\beta}\in\mathbb{F}_1\mathbb{F}_1$$
$$\overline{\overline{\mathsf{A}}} = \sum\mathbf{a}_i\boldsymbol{\alpha}_i = \mathbf{a}_1\boldsymbol{\alpha}_1 + \mathbf{a}_2\boldsymbol{\alpha}_2 + \cdots + \mathbf{a}_n\boldsymbol{\alpha}_n$$
$$\overline{\overline{\mathsf{A}}}|\mathbf{a} = \mathbf{a}|\overline{\overline{\mathsf{A}}}^T, \qquad \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}} = \sum\mathbf{a}_i\boldsymbol{\alpha}_i|\sum\mathbf{b}_j\boldsymbol{\beta}_j = \sum_{i,j}(\boldsymbol{\alpha}_i|\mathbf{b}_j)\mathbf{a}_i\boldsymbol{\beta}_j$$
$$\bar{\bar{\mathsf{I}}} = \sum\mathbf{e}_i\boldsymbol{\varepsilon}_i, \qquad \bar{\bar{\mathsf{I}}}|\overline{\overline{\mathsf{A}}} = \overline{\overline{\mathsf{A}}}|\bar{\bar{\mathsf{I}}} = \overline{\overline{\mathsf{A}}}$$
$$(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}})|\overline{\overline{\mathsf{C}}} = \overline{\overline{\mathsf{A}}}|(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{C}}}), \quad \overline{\overline{\mathsf{A}}}^p = \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{A}}}^{p-1}, \qquad \overline{\overline{\mathsf{A}}}^0 = \bar{\bar{\mathsf{I}}}$$

Double-bar products

$$(\mathbf{a}\boldsymbol{\alpha})||(\mathbf{b}\boldsymbol{\beta})^T = (\mathbf{a}\boldsymbol{\alpha})||(\boldsymbol{\beta}\mathbf{b}) = (\mathbf{a}|\boldsymbol{\beta})(\mathbf{b}|\boldsymbol{\alpha})$$
$$\mathrm{tr}\,\overline{\overline{\mathsf{A}}} = \mathrm{tr}\sum\mathbf{a}_i\boldsymbol{\alpha}_i = \sum\mathbf{a}_i|\boldsymbol{\alpha}_i = \overline{\overline{\mathsf{A}}}||\bar{\bar{\mathsf{I}}}^T$$
$$\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{B}}}^T = \overline{\overline{\mathsf{B}}}||\overline{\overline{\mathsf{A}}}^T = \mathrm{tr}(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}}) = (\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}})||\bar{\bar{\mathsf{I}}}^T$$

Double-wedge products

$$(\mathbf{a}\boldsymbol{\alpha}){}^\wedge_\wedge(\mathbf{b}\boldsymbol{\beta}) = (\mathbf{a}\wedge\mathbf{b})(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}), \qquad (\mathbf{a}\mathbf{b}){}^\wedge_\wedge(\mathbf{c}\mathbf{d}) = (\mathbf{a}\wedge\mathbf{c})(\mathbf{b}\wedge\mathbf{d})$$

$$\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}} = \overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}, \qquad (\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})^T = \overline{\overline{\mathsf{A}}}^T{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}^T$$

$$(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}){}^\wedge_\wedge\overline{\overline{\mathsf{C}}} = \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}}) = \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}} = \overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}} = \cdots$$

$$(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})|(\mathbf{a}\wedge\mathbf{b}) = (\overline{\overline{\mathsf{A}}}|\mathbf{a})\wedge(\overline{\overline{\mathsf{B}}}|\mathbf{b}) + (\overline{\overline{\mathsf{B}}}|\mathbf{a})\wedge(\overline{\overline{\mathsf{A}}}|\mathbf{b})$$

$$(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})\lfloor\mathbf{a} = (\overline{\overline{\mathsf{A}}}|\mathbf{a})\wedge\overline{\overline{\mathsf{B}}} + (\overline{\overline{\mathsf{B}}}|\mathbf{a})\wedge\overline{\overline{\mathsf{A}}}$$

$$\mathbf{a}\rfloor(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}) = \overline{\overline{\mathsf{A}}}\wedge(\mathbf{a}|\overline{\overline{\mathsf{B}}}) + \overline{\overline{\mathsf{B}}}\wedge(\mathbf{a}|\overline{\overline{\mathsf{A}}})$$

Double-wedge powers

$$\overline{\overline{\mathsf{A}}}^{(2)} = \frac{1}{2}\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}} = \frac{1}{2}\sum\mathbf{a}_i\boldsymbol{\alpha}_i{}^\wedge_\wedge\sum\mathbf{a}_j\boldsymbol{\alpha}_j = \sum_{i<j}(\mathbf{a}_i\wedge\mathbf{a}_j)(\boldsymbol{\alpha}_i\wedge\boldsymbol{\alpha}_j) = \sum_{i<j}\mathbf{a}_{ij}\boldsymbol{\alpha}_{ij}$$

$$\overline{\overline{\mathsf{A}}}^{(p)} = \frac{1}{p!}\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\cdots{}^\wedge_\wedge\overline{\overline{\mathsf{A}}} = \frac{r!s!}{p!}\overline{\overline{\mathsf{A}}}^{(r)}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}^{(s)}, \qquad r+s=p$$

$$\overline{\overline{\mathsf{A}}}^{(n)} = \mathbf{a}_N\boldsymbol{\alpha}_N = \mathbf{a}_{123\cdots n}\boldsymbol{\alpha}_{123\cdots n} = (\mathbf{a}_N|\boldsymbol{\alpha}_N)\mathbf{e}_N\boldsymbol{\varepsilon}_N = (\det\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{I}}}^{(n)}$$

$$(\overline{\overline{\mathsf{A}}}|\mathbf{a})\wedge(\overline{\overline{\mathsf{A}}}|\mathbf{b}) = \overline{\overline{\mathsf{A}}}^{(2)}|(\mathbf{a}\wedge\mathbf{b})$$

$$(\overline{\overline{\mathsf{A}}}|\mathbf{a}_1)\wedge(\overline{\overline{\mathsf{A}}}|\mathbf{a}_2)\cdots\wedge(\overline{\overline{\mathsf{A}}}|\mathbf{a}_p) = \overline{\overline{\mathsf{A}}}^{(p)}|(\mathbf{a}_1\wedge\mathbf{a}_2\wedge\cdots\wedge\mathbf{a}_p)$$

$$\overline{\overline{\mathsf{A}}}^{(2)}\lfloor\mathbf{a} = (\overline{\overline{\mathsf{A}}}|\mathbf{a})\wedge\overline{\overline{\mathsf{A}}}, \qquad \boldsymbol{\alpha}\rfloor\overline{\overline{\mathsf{A}}}^{(2)} = \overline{\overline{\mathsf{A}}}\wedge(\boldsymbol{\alpha}|\overline{\overline{\mathsf{A}}})$$

$$\overline{\overline{\mathsf{A}}}^{(3)}\lfloor(\mathbf{a}\wedge\mathbf{b}) = (\overline{\overline{\mathsf{A}}}|\mathbf{a})\wedge(\overline{\overline{\mathsf{A}}}|\mathbf{b})\wedge\overline{\overline{\mathsf{A}}} = (\overline{\overline{\mathsf{A}}}^{(2)}|(\mathbf{a}\wedge\mathbf{b}))\wedge\overline{\overline{\mathsf{A}}}$$

$$(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}})^{(p)} = \overline{\overline{\mathsf{A}}}^{(p)}|\overline{\overline{\mathsf{B}}}^{(p)}, \qquad (\overline{\overline{\mathsf{A}}}^q)^{(p)} = (\overline{\overline{\mathsf{A}}}^{(p)})^q$$

$$(\frac{1}{2}\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})^{(2)} = \frac{1}{2}\overline{\overline{\mathsf{A}}}^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}^{(2)}$$

$$n=4, \qquad (\overline{\overline{\mathsf{A}}}^{(2)})^{(2)} = \frac{1}{8}\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}} = 3(\det\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{I}}}^{(4)}$$

$$n=4, \qquad (\overline{\overline{\mathsf{I}}}\rfloor\mathbf{A})^{(2)} = (\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T)^{(2)} = \mathbf{A}\mathbf{A} - \frac{1}{2}(\mathbf{A}\wedge\mathbf{A})\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}, \quad \mathbf{A}\in\mathbb{E}_2$$

Unit dyadics

$$\bar{\bar{\mathsf{I}}}^{(2)} = (\sum \mathbf{e}_i \boldsymbol{\varepsilon}_i)^{(2)} = \sum_{i<j} \mathbf{e}_{ij} \boldsymbol{\varepsilon}_{ij}$$

$$\bar{\bar{\mathsf{I}}}^{(p)} = (\sum \mathbf{e}_i \boldsymbol{\varepsilon}_i)^{(p)} = \sum \mathbf{e}_J \boldsymbol{\varepsilon}_J, \quad J = \{i_1 i_2 \cdots i_p\}, \quad i_1 < i_2 \cdots < i_p$$

$$\bar{\bar{\mathsf{I}}}^{(n)} = \mathbf{e}_N \boldsymbol{\varepsilon}_N = \frac{\mathbf{k}_N \boldsymbol{\kappa}_N}{\mathbf{k}_N | \boldsymbol{\kappa}_N}$$

$$\bar{\bar{\mathsf{I}}}^{(2)} | (\mathbf{a} \wedge \mathbf{b}) = (\bar{\bar{\mathsf{I}}} | \mathbf{a}) \wedge (\bar{\bar{\mathsf{I}}} | \mathbf{b}) = \mathbf{a} \wedge \mathbf{b}$$

$$\bar{\bar{\mathsf{I}}}^{(p)} | (\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_p) = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \mathbf{a}_p$$

$$\mathrm{tr}(\bar{\bar{\mathsf{I}}}^{(p)}) = \bar{\bar{\mathsf{I}}}^{(p)} || \bar{\bar{\mathsf{I}}}^{(p)T} = C_p^n = \frac{n!}{p!(n-p)!}$$

$$\mathrm{tr}(\bar{\bar{\mathsf{A}}} {}^\wedge_\wedge \bar{\bar{\mathsf{B}}}) = (\bar{\bar{\mathsf{A}}} {}^\wedge_\wedge \bar{\bar{\mathsf{B}}}) || \bar{\bar{\mathsf{I}}}^{(2)T}$$

$$\bar{\bar{\mathsf{I}}} {}^\wedge_\wedge \bar{\bar{\mathsf{A}}} = (\mathrm{tr} \bar{\bar{\mathsf{A}}}) \bar{\bar{\mathsf{I}}}^{(2)} - \bar{\bar{\mathsf{I}}}^{(3)} \lfloor\lfloor \bar{\bar{\mathsf{A}}}^T$$

$$\bar{\bar{\mathsf{I}}}^{(n)} \lfloor\lfloor \bar{\bar{\mathsf{I}}}^{(n-p)T} = \bar{\bar{\mathsf{I}}}^{(p)}, \quad 0 < p < n$$

$$\bar{\bar{\mathsf{I}}}^{(p)} \lfloor\lfloor \bar{\bar{\mathsf{I}}}^{(q)T} = \frac{(n-p+q)!}{q!(n-p)!} \bar{\bar{\mathsf{I}}}^{(p-q)}, \quad q < p \le n$$

$$n = 3, \quad \bar{\bar{\mathsf{I}}}^{(2)} = \mathbf{e}_{12} \boldsymbol{\varepsilon}_{12} + \mathbf{e}_{23} \boldsymbol{\varepsilon}_{23} + \mathbf{e}_{31} \boldsymbol{\varepsilon}_{31}, \quad \bar{\bar{\mathsf{I}}}^{(3)} = \mathbf{e}_{123} \boldsymbol{\varepsilon}_{123}$$

$$n = 4, \quad \bar{\bar{\mathsf{I}}}^{(2)} = \mathbf{e}_{12} \boldsymbol{\varepsilon}_{12} + \mathbf{e}_{23} \boldsymbol{\varepsilon}_{23} + \mathbf{e}_{31} \boldsymbol{\varepsilon}_{31} + (\mathbf{e}_1 \boldsymbol{\varepsilon}_1 + \mathbf{e}_2 \boldsymbol{\varepsilon}_2 + \mathbf{e}_3 \boldsymbol{\varepsilon}_3) {}^\wedge_\wedge \mathbf{e}_4 \boldsymbol{\varepsilon}_4$$

$$n = 4, \quad \bar{\bar{\mathsf{I}}}^{(3)} = \mathbf{e}_{123} \boldsymbol{\varepsilon}_{123} + (\mathbf{e}_{12} \boldsymbol{\varepsilon}_{12} + \mathbf{e}_{23} \boldsymbol{\varepsilon}_{23} + \mathbf{e}_{31} \boldsymbol{\varepsilon}_{31}) {}^\wedge_\wedge \mathbf{e}_4 \boldsymbol{\varepsilon}_4$$

$$n = 4, \quad \mathbf{e}_N \lfloor \bar{\bar{\mathsf{I}}}^{(2)T} = \bar{\bar{\mathsf{I}}}^{(2)} \rfloor \mathbf{e}_N = (\mathbf{e}_N \lfloor \bar{\bar{\mathsf{I}}}^{(2)T})^T = (\bar{\bar{\mathsf{I}}}^{(2)} \rfloor \mathbf{e}_N)^T$$

$$= \mathbf{e}_{12} \mathbf{e}_{34} + \mathbf{e}_{23} \mathbf{e}_{14} + \mathbf{e}_{31} \mathbf{e}_{24} + \mathbf{e}_{14} \mathbf{e}_{23} + \mathbf{e}_{24} \mathbf{e}_{31} + \mathbf{e}_{34} \mathbf{e}_{12}$$

$$n = 4, \quad \bar{\bar{\mathsf{I}}}^{(2)} \lfloor\lfloor \bar{\bar{\mathsf{I}}}^T = 3 \bar{\bar{\mathsf{I}}}, \quad \bar{\bar{\mathsf{I}}}^{(3)} \lfloor\lfloor \bar{\bar{\mathsf{I}}}^T = 2 \bar{\bar{\mathsf{I}}}^{(2)}, \quad \bar{\bar{\mathsf{I}}}^{(3)} \lfloor\lfloor \bar{\bar{\mathsf{I}}}^{(2)T} = 3 \bar{\bar{\mathsf{I}}}$$

$$n = 4, \quad \bar{\bar{\mathsf{I}}}^{(4)} \lfloor\lfloor \bar{\bar{\mathsf{I}}}^T = \bar{\bar{\mathsf{I}}}^{(3)}, \quad \bar{\bar{\mathsf{I}}}^{(4)} \lfloor\lfloor \bar{\bar{\mathsf{I}}}^{(2)T} = \bar{\bar{\mathsf{I}}}^{(2)}, \quad \bar{\bar{\mathsf{I}}}^{(4)} \lfloor\lfloor \bar{\bar{\mathsf{I}}}^{(3)T} = 3 \bar{\bar{\mathsf{I}}}$$

Multivectors and unit dyadics

$$(\mathbf{a} \wedge \mathbf{b}) \lfloor \bar{\bar{\mathsf{I}}}^T = \mathbf{ba} - \mathbf{ab} = -\bar{\bar{\mathsf{I}}} \rfloor (\mathbf{a} \wedge \mathbf{b}) = (\bar{\bar{\mathsf{I}}} \rfloor (\mathbf{a} \wedge \mathbf{b}))^T$$

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \lfloor \bar{\bar{\mathsf{I}}}^T = (\mathbf{a} \wedge \mathbf{b}) \mathbf{c} + (\mathbf{b} \wedge \mathbf{c}) \mathbf{a} + (\mathbf{c} \wedge \mathbf{a}) \mathbf{b} = \bar{\bar{\mathsf{I}}}^{(2)} \rfloor (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$$

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \lfloor \bar{\bar{\mathsf{I}}}^{(2)T} = \mathbf{a} (\mathbf{b} \wedge \mathbf{c}) + \mathbf{b} (\mathbf{c} \wedge \mathbf{a}) + \mathbf{c} (\mathbf{a} \wedge \mathbf{b}) = \bar{\bar{\mathsf{I}}} \rfloor (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})$$

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d}) \lfloor \bar{\bar{\mathsf{I}}}^T = -\bar{\bar{\mathsf{I}}}^{(3)} \rfloor (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d})$$

$$= \mathbf{d} \wedge ((\mathbf{a} \wedge \mathbf{b}) \mathbf{c} + (\mathbf{b} \wedge \mathbf{c}) \mathbf{a} + (\mathbf{c} \wedge \mathbf{a}) \mathbf{b}) - (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \mathbf{d}$$

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d}) \lfloor \bar{\bar{\mathsf{I}}}^{(2)T} = (\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d}) + (\mathbf{c} \wedge \mathbf{a})(\mathbf{b} \wedge \mathbf{d}) + (\mathbf{b} \wedge \mathbf{c})(\mathbf{a} \wedge \mathbf{d})$$

$$+ (\mathbf{a} \wedge \mathbf{d})(\mathbf{b} \wedge \mathbf{c}) + (\mathbf{b} \wedge \mathbf{d})(\mathbf{c} \wedge \mathbf{a}) + (\mathbf{c} \wedge \mathbf{d})(\mathbf{a} \wedge \mathbf{b}) = \bar{\bar{\mathsf{I}}}^{(2)} \rfloor (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d})$$

$$= \{(\mathbf{b}\wedge\mathbf{c})\mathbf{a} + (\mathbf{c}\wedge\mathbf{a})\mathbf{b} + (\mathbf{a}\wedge\mathbf{b})\mathbf{c}\}\wedge\mathbf{d} - \mathbf{d}\wedge\{\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b})\}$$

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})\lfloor\overline{\overline{\mathsf{I}}}^{(3)T} = -\overline{\overline{\mathsf{I}}}\rfloor(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})$$

$$= (\mathbf{a}(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}(\mathbf{a}\wedge\mathbf{b}))\wedge\mathbf{d} - \mathbf{d}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})$$

$$\overline{\overline{\mathsf{I}}}^{(2)}\rfloor\mathbf{e}_N = \sum \mathbf{e}_{ij}\boldsymbol{\varepsilon}_{ij}\rfloor\mathbf{e}_N = \sum(-1)^{i+j-1}\mathbf{e}_{ij}\mathbf{e}_{K(ij)}, \quad i<j$$

$$(\mathbf{A}\wedge\mathbf{B})\lfloor\overline{\overline{\mathsf{I}}}^T = \mathbf{A}\wedge\overline{\overline{\mathsf{I}}}\rfloor\mathbf{B} + \mathbf{B}\wedge\overline{\overline{\mathsf{I}}}\rfloor\mathbf{A} = -\overline{\overline{\mathsf{I}}}^{(3)}\rfloor(\mathbf{A}\wedge\mathbf{B}), \qquad \mathbf{A},\mathbf{B}\in\mathbb{E}_2$$

$$\overline{\overline{\mathsf{I}}}\rfloor(\mathbf{A}\wedge\mathbf{B}) = \mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T\wedge\mathbf{B} + \mathbf{B}\lfloor\overline{\overline{\mathsf{I}}}^T\wedge\mathbf{A} = -(\mathbf{A}\wedge\mathbf{B})\rfloor\overline{\overline{\mathsf{I}}}^{(3)T}$$

$$(\mathbf{A}\wedge\mathbf{B})\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} = \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} - (\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T){}^{\wedge}_{\wedge}(\mathbf{B}\lfloor\overline{\overline{\mathsf{I}}}^T) = \overline{\overline{\mathsf{I}}}^{(2)}\rfloor(\mathbf{A}\wedge\mathbf{B})$$

$$\frac{1}{2}(\mathbf{A}\wedge\mathbf{A})\lfloor\overline{\overline{\mathsf{I}}}^{(2)T} = \mathbf{A}\mathbf{A} - (\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T)^{(2)} = \frac{1}{2}\overline{\overline{\mathsf{I}}}^{(2)}\rfloor(\mathbf{A}\wedge\mathbf{A})$$

$$\overline{\overline{\mathsf{A}}}^T|(\boldsymbol{\Phi}\lfloor\overline{\overline{\mathsf{A}}}) = \boldsymbol{\Phi}|\overline{\overline{\mathsf{A}}}^{(2)}\lfloor\overline{\overline{\mathsf{I}}}^T, \quad \overline{\overline{\mathsf{A}}}\in\mathbb{E}_1\mathbb{F}_1,\ \boldsymbol{\Phi}\in\mathbb{F}_2$$

$$\mathbf{a}\wedge(\mathbf{k}_N\lfloor\overline{\overline{\mathsf{I}}}^T) = \mathbf{k}_N\mathbf{a}, \qquad (\,\overline{\overline{\mathsf{I}}}\rfloor\mathbf{k}_N)\wedge\mathbf{a} = \mathbf{a}\mathbf{k}_N \qquad \mathbf{a}\in\mathbb{E}_1$$

$$\mathbf{A}\wedge(\mathbf{k}_N\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}) = \mathbf{k}_N\mathbf{A}, \qquad (\,\overline{\overline{\mathsf{I}}}^{(2)}\rfloor\mathbf{k}_N)\wedge\mathbf{A} = \mathbf{A}\mathbf{k}_N, \qquad \mathbf{A}\in\mathbb{E}_2$$

$$n=3, \qquad \mathbf{A}\wedge\overline{\overline{\mathsf{I}}}\rfloor\mathbf{B} + \mathbf{B}\wedge\overline{\overline{\mathsf{I}}}\rfloor\mathbf{A} = 0, \quad \mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T\wedge\mathbf{A} = 0$$

$$n=3, \qquad (\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T){}^{\wedge}_{\wedge}(\mathbf{B}\lfloor\overline{\overline{\mathsf{I}}}^T) = \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A}, \qquad (\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T)^{(2)} = \mathbf{A}\mathbf{A}$$

$$n=4, \qquad (\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T){}^{\wedge}_{\wedge}(\mathbf{B}\lfloor\overline{\overline{\mathsf{I}}}^T) = \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A} - (\mathbf{A}\wedge\mathbf{B})\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}$$

$$n=4, \qquad (\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T\wedge\mathbf{B} + \mathbf{B}\lfloor\overline{\overline{\mathsf{I}}}^T\wedge\mathbf{A})\wedge\mathbf{a} = \mathbf{a}(\mathbf{A}\wedge\mathbf{B})$$

$$n=4, \quad (\boldsymbol{\varepsilon}_N\lfloor\mathbf{A})\lfloor(\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T) = -\boldsymbol{\varepsilon}_N|(\mathbf{A}\wedge\mathbf{A})\overline{\overline{\mathsf{I}}}^T$$

Double multiplications

$$\mathrm{tr}(\overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}) = (\mathrm{tr}\overline{\overline{\mathsf{A}}})(\mathrm{tr}\overline{\overline{\mathsf{B}}}) - \mathrm{tr}(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}})$$

$$(\overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}})|(\overline{\overline{\mathsf{C}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{D}}}) = (\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{C}}}){}^{\wedge}_{\wedge}(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{D}}}) + (\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{D}}}){}^{\wedge}_{\wedge}(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{C}}})$$

$$\mathrm{tr}((\overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}})|(\overline{\overline{\mathsf{C}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{D}}})) = (\overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}})||(\overline{\overline{\mathsf{C}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{D}}})^T =$$

$$= (\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{C}}}^T)(\overline{\overline{\mathsf{B}}}||\overline{\overline{\mathsf{D}}}^T) + (\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{D}}}^T)(\overline{\overline{\mathsf{B}}}||\overline{\overline{\mathsf{C}}}^T) - (\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{D}}})||(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{C}}})^T - (\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{C}}})||(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{D}}})^T$$

$$(\overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}})\lfloor\lfloor\overline{\overline{\mathsf{C}}}^T = (\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{C}}}^T)\overline{\overline{\mathsf{B}}} + (\overline{\overline{\mathsf{B}}}||\overline{\overline{\mathsf{C}}}^T)\overline{\overline{\mathsf{A}}} - \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{B}}} - \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{A}}}$$

$$\overline{\overline{\mathsf{A}}}^{(2)}\lfloor\lfloor\overline{\overline{\mathsf{B}}}^T = (\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{B}}}^T)\overline{\overline{\mathsf{A}}} - \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{A}}}$$

$$\overline{\overline{\mathsf{A}}}^{(2)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = (\mathrm{tr}\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{A}}} - \overline{\overline{\mathsf{A}}}^2, \qquad \overline{\overline{\mathsf{I}}}^{(2)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T = (\mathrm{tr}\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{I}}} - \overline{\overline{\mathsf{A}}}$$

$$(\overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{C}}})\lfloor\lfloor\overline{\overline{\mathsf{D}}}^T = (\overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{B}}})(\overline{\overline{\mathsf{C}}}||\overline{\overline{\mathsf{D}}}^T) + (\overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{C}}})(\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{D}}}^T) + (\overline{\overline{\mathsf{C}}}{}^{\wedge}_{\wedge}\overline{\overline{\mathsf{A}}})(\overline{\overline{\mathsf{B}}}||\overline{\overline{\mathsf{D}}}^T)$$

$$-\overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{C}}} + \overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{B}}}) - \overline{\overline{\mathsf{B}}}{}^{\wedge}_{\wedge}(\overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{A}}} + \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{C}}}) - \overline{\overline{\mathsf{C}}}{}^{\wedge}_{\wedge}(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{B}}} + \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{A}}})$$

$$\overline{\overline{\mathsf{A}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{B}}}^T = (\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{B}}}^T)\overline{\overline{\mathsf{A}}}^{(2)} - \overline{\overline{\mathsf{A}}}{}^{\wedge}_{\wedge}(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{A}}})$$

$$\overline{\overline{\mathsf{A}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = (\mathrm{tr}\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{A}}}^{(2)} - \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}^2, \qquad \overline{\overline{\mathsf{I}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T = (\mathrm{tr}\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{I}}}^{(2)} - \overline{\overline{\mathsf{I}}}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}$$

$$\overline{\overline{\mathsf{I}}}^{(3)}\lfloor\lfloor(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})^T = \mathrm{tr}(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})\overline{\overline{\mathsf{I}}} - (\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T$$

$$= (\mathrm{tr}\overline{\overline{\mathsf{A}}})(\mathrm{tr}\overline{\overline{\mathsf{B}}})\overline{\overline{\mathsf{I}}} - \mathrm{tr}(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}})\overline{\overline{\mathsf{I}}} - (\mathrm{tr}\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{B}}} - (\mathrm{tr}\overline{\overline{\mathsf{B}}})\overline{\overline{\mathsf{A}}} + \overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}} + \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{A}}}$$

$$\overline{\overline{\mathsf{I}}}^{(3)}\lfloor\lfloor\overline{\overline{\mathsf{C}}}^T = (\mathrm{tr}\overline{\overline{\mathsf{C}}})\overline{\overline{\mathsf{I}}} - \overline{\overline{\mathsf{C}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T, \qquad \overline{\overline{\mathsf{C}}} \in \mathbb{E}_2\mathbb{F}_2$$

$$(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}}{}^\wedge_\wedge\overline{\overline{\mathsf{D}}})\lfloor\lfloor\overline{\overline{\mathsf{E}}}^T =$$

$$(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}})(\overline{\overline{\mathsf{D}}}||\overline{\overline{\mathsf{E}}}^T) + (\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{D}}})(\overline{\overline{\mathsf{C}}}||\overline{\overline{\mathsf{E}}}^T) + (\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}}{}^\wedge_\wedge\overline{\overline{\mathsf{D}}})(\overline{\overline{\mathsf{B}}}||\overline{\overline{\mathsf{E}}}^T) + (\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}}{}^\wedge_\wedge\overline{\overline{\mathsf{D}}})(\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{E}}}^T)$$

$$-(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}){}^\wedge_\wedge(\overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{D}}} + \overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{C}}}) - (\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}}){}^\wedge_\wedge(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{D}}} + \overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{B}}}) - (\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}){}^\wedge_\wedge(\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{C}}} + \overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{B}}})$$

$$-(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}}){}^\wedge_\wedge(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{D}}} + \overline{\overline{\mathsf{D}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{A}}}) - (\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}){}^\wedge_\wedge(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{C}}} + \overline{\overline{\mathsf{C}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{A}}}) - (\overline{\overline{\mathsf{C}}}{}^\wedge_\wedge\overline{\overline{\mathsf{D}}}){}^\wedge_\wedge(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{B}}} + \overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{E}}}|\overline{\overline{\mathsf{A}}})$$

$$\overline{\overline{\mathsf{A}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{B}}}^T = (\overline{\overline{\mathsf{A}}}||\overline{\overline{\mathsf{B}}}^T)\overline{\overline{\mathsf{A}}}^{(3)} - \overline{\overline{\mathsf{A}}}^{(2)}{}^\wedge_\wedge(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}}|\overline{\overline{\mathsf{A}}})$$

$$\overline{\overline{\mathsf{A}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T = (\mathrm{tr}\,\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{A}}}^{(3)} - \overline{\overline{\mathsf{A}}}^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}^2, \qquad \overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^T = (\mathrm{tr}\,\overline{\overline{\mathsf{A}}})\overline{\overline{\mathsf{I}}}^{(3)} - \overline{\overline{\mathsf{I}}}^{(2)}{}^\wedge_\wedge\overline{\overline{\mathsf{A}}}$$

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})^T = \mathrm{tr}(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})\overline{\overline{\mathsf{I}}}^{(2)} - ((\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}})\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} + \overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}$$

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{C}}}^T = (\mathrm{tr}\overline{\overline{\mathsf{C}}})\overline{\overline{\mathsf{I}}}^{(2)} - (\overline{\overline{\mathsf{C}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^T){}^\wedge_\wedge\overline{\overline{\mathsf{I}}} + \overline{\overline{\mathsf{C}}}, \qquad \overline{\overline{\mathsf{C}}} \in \mathbb{E}_2\mathbb{F}_2$$

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}})^T = \mathrm{tr}(\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}})\overline{\overline{\mathsf{I}}} - (\overline{\overline{\mathsf{A}}}{}^\wedge_\wedge\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge\overline{\overline{\mathsf{C}}})\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}$$

$$\overline{\overline{\mathsf{I}}}^{(4)}\lfloor\lfloor\overline{\overline{\mathsf{D}}}^T = (\mathrm{tr}\overline{\overline{\mathsf{D}}})\overline{\overline{\mathsf{I}}} - \overline{\overline{\mathsf{D}}}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(2)T}, \qquad \overline{\overline{\mathsf{D}}} \in \mathbb{E}_3\mathbb{F}_3$$

$$\overline{\overline{\mathsf{A}}}^{(p+1)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{-1T} = (n-p)\overline{\overline{\mathsf{A}}}^{(p)}, \quad \overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{F}_1$$

$$\overline{\overline{\mathsf{A}}}^{(p)}{}^\wedge_\wedge\overline{\overline{\mathsf{I}}}^{(n-p)} = (\mathrm{tr}\,\overline{\overline{\mathsf{A}}}^{(p)})\,\overline{\overline{\mathsf{I}}}^{(n)}$$

$$\overline{\overline{\mathsf{A}}}^{(p)}\lfloor\lfloor\overline{\overline{\mathsf{I}}}^{(p-1)T} = (\overline{\overline{\mathsf{I}}}^{(p)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(p-1)T})|\overline{\overline{\mathsf{A}}}, \qquad p > 1$$

$$\overline{\overline{\mathsf{I}}}^{(p+1)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(p)T} = (\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(p)})\overline{\overline{\mathsf{I}}} - (\overline{\overline{\mathsf{I}}}^{(p)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(p-1)T})|\overline{\overline{\mathsf{A}}}, \quad p > 1$$

$$= (\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(p)})\overline{\overline{\mathsf{I}}} - (\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(p-1)})\overline{\overline{\mathsf{A}}} + (\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(p-2)})\overline{\overline{\mathsf{A}}}^2 - \cdots + (-\overline{\overline{\mathsf{A}}})^p$$

$$0 = (\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(n)})\overline{\overline{\mathsf{I}}} - (\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(n-1)})\overline{\overline{\mathsf{A}}} + (\mathrm{tr}\overline{\overline{\mathsf{A}}}^{(n-2)})\overline{\overline{\mathsf{A}}}^2 - \cdots + (-\overline{\overline{\mathsf{A}}})^n$$

Inverse dyadics

$$(\overline{\overline{\mathsf{A}}}|\overline{\overline{\mathsf{B}}})^{-1} = \overline{\overline{\mathsf{B}}}^{-1}|\overline{\overline{\mathsf{A}}}^{-1}, \quad \mathrm{det}\overline{\overline{\mathsf{A}}} = \mathrm{tr}\overline{\overline{\mathsf{A}}}^{(n)}$$

$$(\overline{\overline{\mathsf{A}}}^{(p)})^{-1} = (\overline{\overline{\mathsf{A}}}^{-1})^{(p)} = \ (\mathrm{def})\ \overline{\overline{\mathsf{A}}}^{(-p)}, \quad 1 < p < n,$$

$$\overline{\overline{\mathsf{A}}}^{-1} = \frac{\overline{\overline{\mathsf{I}}}^{(n)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(n-1)T}}{\mathrm{det}\overline{\overline{\mathsf{A}}}}, \quad \overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{F}_1$$

$$\overline{\overline{\mathsf{A}}}^{-1} = \frac{\boldsymbol{\kappa}_N\boldsymbol{\kappa}_N\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(n-1)T}}{\boldsymbol{\kappa}_N\boldsymbol{\kappa}_N||\overline{\overline{\mathsf{A}}}^{(n)}}, \quad \overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{E}_1$$

$$\overline{\overline{\mathsf{A}}}^{(-p)} = \frac{\overline{\overline{\mathsf{I}}}^{(n)}\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(n-p)T}}{\mathrm{det}\ \overline{\overline{\mathsf{A}}}}, \quad \overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{F}_1$$

$$\overline{\overline{\mathsf{A}}}^{(-p)} = \frac{\boldsymbol{\kappa}_N\boldsymbol{\kappa}_N\lfloor\lfloor\overline{\overline{\mathsf{A}}}^{(n-p)T}}{\boldsymbol{\kappa}_N\boldsymbol{\kappa}_N||\overline{\overline{\mathsf{A}}}^{(n)}}, \quad \overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{E}_1$$

$$(\mathbf{e}_N\lfloor\overline{\overline{\mathsf{I}}}^{(p)T})^{-1} = (-1)^{p(n-p)}\boldsymbol{\varepsilon}_N\lfloor\overline{\overline{\mathsf{I}}}^{(n-p)}$$

$$(\overline{\overline{\mathsf{A}}} + \mathbf{a}\boldsymbol{\alpha})^{-1} = \overline{\overline{\mathsf{A}}}^{-1} - \frac{\overline{\overline{\mathsf{A}}}^{-1}|\mathbf{a}\boldsymbol{\alpha}|\overline{\overline{\mathsf{A}}}^{-1}}{1 + \boldsymbol{\alpha}|\overline{\overline{\mathsf{A}}}^{-1}|\mathbf{a}}, \quad \overline{\overline{\mathsf{A}}} \in \mathbb{E}_1\mathbb{F}_1$$

$$(\overline{\overline{\mathsf{C}}} + \mathbf{AB})^{-1} = \overline{\overline{\mathsf{C}}}^{-1} - \frac{\overline{\overline{\mathsf{C}}}^{-1}|\mathbf{AB}|\overline{\overline{\mathsf{C}}}^{-1}}{1 + \mathbf{B}|\overline{\overline{\mathsf{C}}}^{-1}|\mathbf{A}}, \quad \overline{\overline{\mathsf{C}}} \in \mathbb{E}_2\mathbb{E}_2$$

$$n = 4, \quad (\mathbf{A}\lfloor\overline{\overline{\mathsf{I}}}^T)^{-1} = -2(\boldsymbol{\varepsilon}_N\lfloor\mathbf{A})\lfloor\overline{\overline{\mathsf{I}}}\ /(\boldsymbol{\varepsilon}_N|(\mathbf{A}\wedge\mathbf{A}))$$

$$n = 4, \quad (\overline{\overline{\mathsf{A}}} + \mathbf{e}_4\mathbf{a} + \mathbf{b}\mathbf{e}_4 + c\mathbf{e}_4\mathbf{e}_4)^{-1} = \overline{\overline{\mathsf{A}}}^{-1} + \frac{(\overline{\overline{\mathsf{A}}}^{-1}|\mathbf{b} - \boldsymbol{\varepsilon}_4)(\mathbf{a}|\overline{\overline{\mathsf{A}}}^{-1} - \boldsymbol{\varepsilon}_4)}{c - \mathbf{a}|\overline{\overline{\mathsf{A}}}^{-1}|\mathbf{b}}$$

$$n = 4, \quad (\overline{\overline{\mathsf{A}}} + \mathbf{e}_4\boldsymbol{\alpha} + \mathbf{b}\boldsymbol{\varepsilon}_4 + c\mathbf{e}_4\boldsymbol{\varepsilon}_4)^{-1} = \overline{\overline{\mathsf{A}}}^{-1} + \frac{(\overline{\overline{\mathsf{A}}}^{-1}|\mathbf{b} - \mathbf{e}_4)(\boldsymbol{\alpha}|\overline{\overline{\mathsf{A}}}^{-1} - \boldsymbol{\varepsilon}_4)}{c - \boldsymbol{\alpha}|\overline{\overline{\mathsf{A}}}^{-1}|\mathbf{b}}$$

$$n = 4, \quad \mathrm{det}(\overline{\overline{\mathsf{A}}} + \mathbf{e}_4\boldsymbol{\alpha} + \mathbf{b}\boldsymbol{\varepsilon}_4 + c\mathbf{e}_4\boldsymbol{\varepsilon}_4) = \mathbf{e}_{123}|(c\overline{\overline{\mathsf{A}}}^{(3)T} - \overline{\overline{\mathsf{A}}}^{(2)T}{}^{\wedge}_{\wedge}\boldsymbol{\alpha}\mathbf{b})|\boldsymbol{\varepsilon}_{123}$$

$$n = 3, \quad \overline{\overline{\mathsf{C}}}^{-1} = \frac{1}{\mathrm{Det}\overline{\overline{\mathsf{C}}}}(\overline{\overline{\mathsf{C}}}^T\rfloor\rfloor\overline{\overline{\mathsf{I}}}^{(3)})^{(2)}, \quad \overline{\overline{\mathsf{C}}} \in \mathbb{E}_2\mathbb{F}_2,$$

$$n = 3, \quad \mathrm{Det}\ \overline{\overline{\mathsf{C}}} = \mathrm{det}(\overline{\overline{\mathsf{C}}}^T\rfloor\rfloor\overline{\overline{\mathsf{I}}}^{(3)}) = (\overline{\overline{\mathsf{C}}}^T\rfloor\rfloor\overline{\overline{\mathsf{I}}}^{(3)})^{(3)}||\overline{\overline{\mathsf{I}}}^{(3)T}, \quad \overline{\overline{\mathsf{C}}} \in \mathbb{E}_2\mathbb{F}_2$$

Metric and Hodge dyadics

3D Euclidean space

$$\overline{\overline{\mathsf{G}}}_{\mathbf{E}} = \mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_3, \quad \overline{\overline{\Gamma}}_{\mathbf{E}} = \overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{-1} = \boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2\boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_3\boldsymbol{\varepsilon}_3$$

$$\overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{(2)} = \mathbf{e}_{12}\mathbf{e}_{12} + \mathbf{e}_{23}\mathbf{e}_{23} + \mathbf{e}_{31}\mathbf{e}_{31}, \quad \overline{\overline{\Gamma}}_{\mathbf{E}}^{(2)} = \boldsymbol{\varepsilon}_{12}\boldsymbol{\varepsilon}_{12} + \boldsymbol{\varepsilon}_{23}\boldsymbol{\varepsilon}_{23} + \boldsymbol{\varepsilon}_{31}\boldsymbol{\varepsilon}_{31}$$

$$\overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{(3)} = \mathbf{e}_{123}\mathbf{e}_{123}, \quad \overline{\overline{\Gamma}}_{\mathbf{E}}^{(3)} = \boldsymbol{\varepsilon}_{123}\boldsymbol{\varepsilon}_{123}$$

$$\overline{\overline{\mathsf{H}}}_{\mathbf{E}1} = \overline{\overline{\mathsf{H}}}_{\mathbf{E}2}^{-1} = \mathbf{e}_{12}\boldsymbol{\varepsilon}_3 + \mathbf{e}_{23}\boldsymbol{\varepsilon}_1 + \mathbf{e}_{31}\boldsymbol{\varepsilon}_2 = \mathbf{e}_{123}\lfloor\overline{\overline{\Gamma}}_{\mathbf{E}} = \overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{(2)}\rfloor\boldsymbol{\varepsilon}_{123}$$

$$\overline{\overline{\mathsf{H}}}_{\mathbf{E}2} = \overline{\overline{\mathsf{H}}}_{\mathbf{E}1}^{-1} = \mathbf{e}_1\boldsymbol{\varepsilon}_{23} + \mathbf{e}_2\boldsymbol{\varepsilon}_{31} + \mathbf{e}_3\boldsymbol{\varepsilon}_{12} = \overline{\overline{\mathsf{G}}}_{\mathbf{E}}\rfloor\boldsymbol{\varepsilon}_{123} = \mathbf{e}_{123}\lfloor\overline{\overline{\Gamma}}_{\mathbf{E}}^{(2)}$$

$$\mathbf{e}_i \wedge \overline{\overline{\mathsf{H}}}_{\mathbf{E}2} = \overline{\overline{\mathsf{H}}}_{\mathbf{E}1} \wedge \boldsymbol{\varepsilon}_i$$

$$\mathbf{a} \wedge \overline{\overline{\mathsf{H}}}_{\mathbf{E}1}|\mathbf{b} = \mathbf{e}_{123}(\mathbf{a}|\overline{\overline{\Gamma}}_{\mathbf{E}}|\mathbf{b}), \quad \mathbf{A} \wedge \overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{(2)}|\mathbf{B} = \mathbf{e}_{123}(\mathbf{A}|\overline{\overline{\Gamma}}_{\mathbf{E}}^{(2)}|\mathbf{B})$$

4D Minkowski space

$$\overline{\overline{\mathsf{G}}}_{\mathbf{M}} = \overline{\overline{\mathsf{G}}}_{\mathbf{E}} - \mathbf{e}_4\mathbf{e}_4, \quad \overline{\overline{\Gamma}}_{\mathbf{M}} = \overline{\overline{\Gamma}}_{\mathbf{E}} - \boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4$$

$$\overline{\overline{\mathsf{G}}}_{\mathbf{M}}^{(2)} = \overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{(2)} - \overline{\overline{\mathsf{G}}}_{\mathbf{E}}{}^{\wedge}_{\wedge}\mathbf{e}_4\mathbf{e}_4, \quad \overline{\overline{\Gamma}}_{\mathbf{M}}^{(2)} = \overline{\overline{\Gamma}}_{\mathbf{E}}^{(2)} - \overline{\overline{\Gamma}}_{\mathbf{E}}{}^{\wedge}_{\wedge}\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4$$

$$\overline{\overline{\mathsf{G}}}_{\mathbf{M}}^{(3)} = \overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{(3)} - \overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{(2)}{}^{\wedge}_{\wedge}\mathbf{e}_4\mathbf{e}_4, \quad \overline{\overline{\Gamma}}_{\mathbf{M}}^{(3)} = \overline{\overline{\Gamma}}_{\mathbf{E}}^{(3)} - \overline{\overline{\Gamma}}_{\mathbf{E}}^{(2)}{}^{\wedge}_{\wedge}\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4$$

$$\overline{\overline{\mathsf{G}}}_{\mathbf{M}}^{(4)} = -\overline{\overline{\mathsf{G}}}_{\mathbf{E}}^{(3)}{}^{\wedge}_{\wedge}\mathbf{e}_4\mathbf{e}_4, \quad \overline{\overline{\Gamma}}_{\mathbf{M}}^{(4)} = -\overline{\overline{\Gamma}}_{\mathbf{E}}^{(3)}{}^{\wedge}_{\wedge}\boldsymbol{\varepsilon}_4\boldsymbol{\varepsilon}_4$$

$$\overline{\overline{\mathsf{H}}}_{\mathbf{M}1} = \mathbf{e}_4 \wedge \overline{\overline{\mathsf{H}}}_{\mathbf{E}1} + \mathbf{e}_{123}\boldsymbol{\varepsilon}_4 = \mathbf{e}_{1234}\lfloor\overline{\overline{\Gamma}}_{\mathbf{M}}$$

$$\overline{\overline{\mathsf{H}}}_{\mathbf{M}2} = -\overline{\overline{\mathsf{H}}}_{\mathbf{M}2}^{-1} = -\overline{\overline{\mathsf{H}}}_{\mathbf{E}1} \wedge \boldsymbol{\varepsilon}_4 - \mathbf{e}_4 \wedge \overline{\overline{\mathsf{H}}}_{\mathbf{E}2} = \mathbf{e}_{1234}\lfloor\overline{\overline{\Gamma}}_{\mathbf{M}}^{(2)}$$

$$\overline{\overline{\mathsf{H}}}_{\mathbf{M}3} = \overline{\overline{\mathsf{H}}}_{\mathbf{M}1}^{-1} = \overline{\overline{\mathsf{H}}}_{\mathbf{E}2} \wedge \boldsymbol{\varepsilon}_4 + \mathbf{e}_4\boldsymbol{\varepsilon}_{123} = \mathbf{e}_{1234}\lfloor\overline{\overline{\Gamma}}_{\mathbf{M}}^{(3)}$$

$$\overline{\overline{\mathsf{H}}}_{\mathbf{M}3}|\overline{\overline{\mathsf{H}}}_{\mathbf{M}1} = \overline{\overline{\mathsf{I}}}, \quad \overline{\overline{\mathsf{H}}}_{\mathbf{M}1}|\overline{\overline{\mathsf{H}}}_{\mathbf{M}3} = \overline{\overline{\mathsf{I}}}^{(3)}$$

Medium dyadics

3D Euclidean space

$$\mathbf{D} = \overline{\overline{\epsilon}}|\mathbf{E} + \overline{\overline{\xi}}|\mathbf{H}, \quad \mathbf{B} = \overline{\overline{\zeta}}|\mathbf{E} + \overline{\overline{\mu}}|\mathbf{H}$$

$$\text{Hodge dyadics} \quad \overline{\overline{\epsilon}},\ \overline{\overline{\mu}},\ \overline{\overline{\xi}}, \overline{\overline{\zeta}} \ \in \mathbb{F}_2\mathbb{E}_1$$

$$\text{Inverses} \quad \overline{\overline{\epsilon}}^{-1},\ \overline{\overline{\mu}}^{-1},\ \overline{\overline{\xi}}^{-1},\ \overline{\overline{\zeta}}^{-1} \ \in \mathbb{F}_1\mathbb{E}_2$$

$$\mathbf{e}_{123}\lfloor\mathbf{D} = \overline{\overline{\epsilon}}_g|\mathbf{E} + \overline{\overline{\xi}}_g|\mathbf{H}, \quad \mathbf{e}_{123}\lfloor\mathbf{B} = \overline{\overline{\zeta}}_g|\mathbf{E} + \overline{\overline{\mu}}_g|\mathbf{H}$$

$$\text{Metric (“Gibbsian”) dyadics} \quad \overline{\overline{\epsilon}}_g,\ \overline{\overline{\mu}}_g,\ \overline{\overline{\xi}}_g,\ \overline{\overline{\zeta}}_g \ \in \mathbb{E}_1\mathbb{E}_1$$

$$\text{Inverses} \quad \overline{\overline{\epsilon}}_g^{-1},\ \overline{\overline{\mu}}_g^{-1},\ \overline{\overline{\xi}}_g^{-1},\ \overline{\overline{\zeta}}_g^{-1} \ \in \mathbb{F}_1\mathbb{F}_1$$

$$\text{Relations} \quad \overline{\overline{\epsilon}}_g = \mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}}, \quad \overline{\overline{\epsilon}} = \boldsymbol{\varepsilon}_{123}\lfloor\overline{\overline{\epsilon}}_g, \quad \text{etc}$$

$$\text{Inverse relations} \quad \overline{\overline{\epsilon}}^{-1} = \overline{\overline{\epsilon}}_g^{-1}\rfloor\mathbf{e}_{123}, \quad \overline{\overline{\epsilon}}_g^{-1} = \overline{\overline{\epsilon}}^{-1}\rfloor\boldsymbol{\varepsilon}_{123}, \quad \text{etc}$$

4D Minkowski space $(N = 1234)$

$$\mathbf{\Psi} = \overline{\overline{\overline{\mathsf{M}}}}|\mathbf{\Phi}, \qquad \mathbf{\Phi} = \overline{\overline{\overline{\mathsf{M}}}}^{-1}|\mathbf{\Psi}$$

$$\text{Hodge dyadics} \quad \overline{\overline{\overline{\mathsf{M}}}}, \overline{\overline{\overline{\mathsf{M}}}}^{-1} \in \mathbb{F}_2\mathbb{E}_2,$$

$$\mathbf{e}_N\lfloor\mathbf{\Psi} = \overline{\overline{\overline{\mathsf{M}}}}_g|\mathbf{\Phi}, \qquad \mathbf{e}_N\lfloor\mathbf{\Phi} = (\mathbf{e}_N\mathbf{e}_N\lfloor\lfloor\overline{\overline{\overline{\mathsf{M}}}}_g^{-1})|\mathbf{\Psi}$$

$$\text{Metric ("modified") dyadics} \quad \overline{\overline{\overline{\mathsf{M}}}}_g,\ (\mathbf{e}_N\mathbf{e}_N\lfloor\lfloor\overline{\overline{\overline{\mathsf{M}}}}_g^{-1}) \in \mathbb{E}_2\mathbb{E}_2,\ \overline{\overline{\overline{\mathsf{M}}}}_g^{-1} \in \mathbb{F}_2\mathbb{F}_2$$

$$\text{Relations} \quad \overline{\overline{\overline{\mathsf{M}}}}_g = \mathbf{e}_N\lfloor\overline{\overline{\overline{\mathsf{M}}}}, \qquad \overline{\overline{\overline{\mathsf{M}}}} = \boldsymbol{\varepsilon}_N\lfloor\overline{\overline{\overline{\mathsf{M}}}}_g$$

$$\text{Inverse relations} \quad \overline{\overline{\overline{\mathsf{M}}}}^{-1} = \overline{\overline{\overline{\mathsf{M}}}}_g^{-1}\rfloor\mathbf{e}_N, \qquad \overline{\overline{\overline{\mathsf{M}}}}_g^{-1} = \overline{\overline{\overline{\mathsf{M}}}}^{-1}\rfloor\boldsymbol{\varepsilon}_N$$

Bi-anisotropic medium

$$\overline{\overline{\overline{\mathsf{M}}}} = \overline{\overline{\alpha}} + \overline{\overline{\epsilon}}' \wedge \mathbf{e}_4 + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mu}}^{-1} + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\beta}} \wedge \mathbf{e}_4$$

$$\overline{\overline{\epsilon}}' = \overline{\overline{\epsilon}} - \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}|\overline{\overline{\zeta}}, \qquad \overline{\overline{\alpha}} = \overline{\overline{\xi}}|\overline{\overline{\mu}}^{-1}, \qquad \overline{\overline{\beta}} = -\overline{\overline{\mu}}^{-1}|\overline{\overline{\zeta}}$$

$$\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}' - \overline{\overline{\alpha}}|\overline{\overline{\mu}}|\overline{\overline{\beta}}, \qquad \overline{\overline{\xi}} = \overline{\overline{\alpha}}|\overline{\overline{\mu}}, \qquad \overline{\overline{\zeta}} = -\overline{\overline{\mu}}|\overline{\overline{\beta}}$$

$$\overline{\overline{\overline{\mathsf{M}}}}_g = \overline{\overline{\epsilon}}_g{}^\wedge_\wedge\mathbf{e}_4\mathbf{e}_4 - (\mathbf{e}_{123}\lfloor\overline{\overline{\mathsf{I}}}^T + \mathbf{e}_4 \wedge \overline{\overline{\xi}}_g)|\overline{\overline{\mu}}_g^{-1}|(\overline{\overline{\mathsf{I}}}\rfloor\mathbf{e}_{123} - \overline{\overline{\zeta}}_g \wedge \mathbf{e}_4)$$

$$= -\mathbf{e}_{123}\mathbf{e}_{123}\lfloor\lfloor\overline{\overline{\mu}}_g^{-1} - \mathbf{e}_4 \wedge \overline{\overline{\xi}}_g|\overline{\overline{\mu}}_g^{-1}\rfloor\mathbf{e}_{123} + \mathbf{e}_{123}\lfloor\overline{\overline{\mu}}_g^{-1}|\overline{\overline{\zeta}}_g \wedge \mathbf{e}_4 + (\overline{\overline{\epsilon}}_g - \overline{\overline{\xi}}_g|\overline{\overline{\mu}}_g^{-1}|\overline{\overline{\zeta}}_g){}^\wedge_\wedge\mathbf{e}_4\mathbf{e}_4$$

$$\overline{\overline{\epsilon}}' = \boldsymbol{\varepsilon}_{123}\lfloor(\overline{\overline{\epsilon}}_g - \overline{\overline{\xi}}_g|\overline{\overline{\mu}}_g^{-1}|\overline{\overline{\zeta}}_g), \qquad \overline{\overline{\mu}} = \boldsymbol{\varepsilon}_{123}\lfloor\overline{\overline{\mu}}_g$$

$$\overline{\overline{\alpha}} = \boldsymbol{\varepsilon}_{123}\lfloor(\overline{\overline{\xi}}_g|\overline{\overline{\mu}}_g^{-1})\rfloor\mathbf{e}_{123}, \qquad \overline{\overline{\beta}} = -\overline{\overline{\mu}}_g^{-1}|\overline{\overline{\zeta}}_g$$

$$\overline{\overline{\overline{\mathsf{M}}}} = \boldsymbol{\varepsilon}_{123}\lfloor\overline{\overline{\epsilon}}_g \wedge \mathbf{e}_4 + (\boldsymbol{\varepsilon}_{123}\lfloor\overline{\overline{\xi}}_g + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mathsf{I}}}^T)|\overline{\overline{\mu}}_g^{-1}|(\overline{\overline{\mathsf{I}}}\rfloor\mathbf{e}_{123} - \overline{\overline{\zeta}}_g \wedge \mathbf{e}_4)$$

$$\overline{\overline{\overline{\mathsf{M}}}}^{-1} = \overline{\overline{\alpha}}_1 + \overline{\overline{\epsilon}}_1' \wedge \mathbf{e}_4 + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\mu}}_1^{-1} + \boldsymbol{\varepsilon}_4 \wedge \overline{\overline{\beta}}_1 \wedge \mathbf{e}_4,$$

$$\overline{\overline{\epsilon}}_1' = -(\overline{\overline{\mu}}^{-1} - \overline{\overline{\beta}}|\overline{\overline{\epsilon}}'^{-1}|\overline{\overline{\alpha}})^{-1},$$

$$\overline{\overline{\mu}}_1^{-1} = -(\overline{\overline{\epsilon}}' - \overline{\overline{\alpha}}|\overline{\overline{\mu}}|\overline{\overline{\beta}})^{-1},$$

$$\overline{\overline{\alpha}}_1 = \overline{\overline{\mu}}|\overline{\overline{\beta}}|\overline{\overline{\mu}}_1^{-1} = (\overline{\overline{\alpha}} - \overline{\overline{\epsilon}}'|\overline{\overline{\beta}}^{-1}|\overline{\overline{\mu}}^{-1})^{-1},$$

$$\overline{\overline{\beta}}_1 = \overline{\overline{\epsilon}}'^{-1}|\overline{\overline{\alpha}}|\overline{\overline{\epsilon}}_1' = (\overline{\overline{\beta}} - \overline{\overline{\mu}}^{-1}|\overline{\overline{\alpha}}^{-1}|\overline{\overline{\epsilon}}')^{-1}$$

$$\mathbf{e}_N\mathbf{e}_N\lfloor\lfloor\overline{\overline{\overline{\mathsf{M}}}}_g^{-1} = -\overline{\overline{\mu}}_g{}^\wedge_\wedge\mathbf{e}_4\mathbf{e}_4 + (\mathbf{e}_{123}\lfloor\overline{\overline{\mathsf{I}}}^T - \mathbf{e}_4 \wedge \overline{\overline{\zeta}}_g)|\overline{\overline{\epsilon}}_g^{-1}|(\overline{\overline{\mathsf{I}}}\rfloor\mathbf{e}_{123} + \overline{\overline{\xi}}_g \wedge \mathbf{e}_4)$$

$$= \mathbf{e}_{123}\mathbf{e}_{123}\lfloor\lfloor\overline{\overline{\epsilon}}_g^{-1} - \mathbf{e}_4 \wedge \overline{\overline{\zeta}}_g|\overline{\overline{\epsilon}}_g^{-1}\rfloor\mathbf{e}_{123} + \mathbf{e}_{123}\lfloor\overline{\overline{\epsilon}}_g^{-1}|\overline{\overline{\xi}}_g \wedge \mathbf{e}_4 - (\overline{\overline{\mu}}_g - \overline{\overline{\zeta}}_g|\overline{\overline{\epsilon}}_g^{-1}|\overline{\overline{\xi}}_g){}^\wedge_\wedge\mathbf{e}_4\mathbf{e}_4$$