S-96.510
Advanced Field Theory
Course for graduate students
Lecture viewgraphs, fall term 2004

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Contents

Foreword

This lecture material contains all viewgraphs associated with the graduate course **S-96.510 Advanced Field Theory** given at the Department of Electrical and Communications Engineering, fall 2004. The course is based on Chapters 1–6 of the book *Methods for Electromagnetic Field Analysis* (Oxford University Press 1992, 2nd edition IEEE Press, 1995, 3rd printing Wiley 2002) by this author. The figures drawn by hand on the blackboard could not be added to the present material.

Otaniemi, September 13 2004 I.V. Lindell

S-96.510 Advanced Field Theory
1. Complex Vectors and Dyadics

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Complex Vectors

- Complex vectors $\mathbf{a} = \mathbf{a}_r + j\mathbf{a}_i$, ($\mathbf{a}_r, \mathbf{a}_i$ real vectors)
- Time-harmonic vectors $\mathbf{A}(t) = \mathbf{A}_1 \cos\omega t + \mathbf{A}_2 \sin\omega t$
- Sense of rotation: $\mathbf{A}_1 \to \mathbf{A}_2$ shortest way
- Correspondence $\mathbf{a} \leftrightarrow \mathbf{A}(t)$ through two mappings
- Mapping $\mathbf{a} \to \mathbf{A}(t)$

$$\mathbf{A}(t) = \Re\{\mathbf{a}e^{j\omega t}\} = \mathbf{a}_r \cos\omega t - \mathbf{a}_i \sin\omega t$$

- Inverse mapping $\mathbf{A}(t) \to \mathbf{a}$

$$\mathbf{a} = \mathbf{a}_r + j\mathbf{a}_i = \mathbf{A}(0) - j\mathbf{A}(\pi/2\omega)$$

Special Complex Vectors

- Correspondence

$$\mathbf{a} = \mathbf{a}_r + j\mathbf{a}_i \;\leftrightarrow\; \mathbf{A}(t) = \mathbf{a}_r \cos\omega t - \mathbf{a}_i \sin\omega t$$

- Circularly polarized (CP) vectors $\mathbf{a}\cdot\mathbf{a} = 0$
- CP implies $\mathbf{a}_r \cdot \mathbf{a}_i = 0$ and $|\mathbf{a}_r| = |\mathbf{a}_i|$
- Linearly polarized (LP) vectors $\mathbf{a}\times\mathbf{a}^* = 0$
- LP implies $\mathbf{a}_r \times \mathbf{a}_i = 0$ (parallel vectors $\mathbf{a}_r, \mathbf{a}_i$)
- Elliptical polarization in general
- $\mathbf{a}$ and $\mathbf{b}$ have same ellipse iff $\mathbf{b} = e^{j\theta}\mathbf{a}$, θ real

$$\mathbf{B}(t) = \Re\{\mathbf{b}e^{j\omega t}\} = \Re\{\mathbf{a}e^{j(\omega t+\theta)}\} = \mathbf{A}(t+\theta/\omega)$$

Axial Representation

- To find axes of the ellipse of a complex vector $\mathbf{a}$ ($\mathbf{a}\cdot\mathbf{a} \neq 0$)
- Solution through another complex vector $\mathbf{b}$

$$\mathbf{b} = \mathbf{b}_r + j\mathbf{b}_i = \frac{|\sqrt{\mathbf{a}\cdot\mathbf{a}}|}{\sqrt{\mathbf{a}\cdot\mathbf{a}}}\mathbf{a} = e^{j\theta}\mathbf{a}$$

- $\mathbf{a}$ and $\mathbf{b}$ have same ellipse, same axes (θ real)

$$\mathbf{b}\cdot\mathbf{b} = \mathbf{b}_r\cdot\mathbf{b}_r + 2j\mathbf{b}_r\cdot\mathbf{b}_i - \mathbf{b}_i\cdot\mathbf{b}_i = |\mathbf{a}\cdot\mathbf{a}| > 0$$

- $\mathbf{b}\cdot\mathbf{b}$ real and positive $\Rightarrow \mathbf{b}_r\cdot\mathbf{b}_i = 0$, $|\mathbf{b}_r| > |\mathbf{b}_i|$
- $\mathbf{b}_r, \mathbf{b}_i$ define the axes of the ellipse of $\mathbf{a}$
- $\mathbf{b}_r$ on major axis, $\mathbf{b}_i$ on minor axis

Helicity Vector 1

- Helicity vector $\mathbf{p}(\mathbf{a})$ ('polarization vector') of $\mathbf{a} = \mathbf{a}_r + j\mathbf{a}_i$

$$\mathbf{p}(\mathbf{a}) = \frac{\mathbf{a} \times \mathbf{a}^*}{j\mathbf{a} \cdot \mathbf{a}^*} = \frac{2\mathbf{a}_i \times \mathbf{a}_r}{|\mathbf{a}_r|^2 + |\mathbf{a}_i|^2}$$

- Properties:
- $\mathbf{p}(\mathbf{a}) = [\mathbf{p}(\mathbf{a})]^*$ is a real vector
- $\mathbf{a} \to \mathbf{a}^*$ changes sense of rotation: $\mathbf{p}(\mathbf{a}) \to \mathbf{p}(\mathbf{a}^*) = -\mathbf{p}(\mathbf{a})$
- Linearly polarized vector $\mathbf{a} \times \mathbf{a}^* = 0 \quad \Rightarrow \quad \mathbf{p}(\mathbf{a}) = 0$
- Circularly polarized vector $\mathbf{a} \cdot \mathbf{a} = 0 \quad \Rightarrow \quad |\mathbf{p}(\mathbf{a})| = 1$
- Elliptically polarized vector $0 < |\mathbf{p}(\mathbf{a})| < 1$

Helicity Vector 2

- More properties:

$$\mathbf{p}(\mathbf{a}) = \frac{\mathbf{a} \times \mathbf{a}^*}{j\mathbf{a} \cdot \mathbf{a}^*} = \frac{2\mathbf{a}_i \times \mathbf{a}_r}{|\mathbf{a}_r|^2 + |\mathbf{a}_i|^2}$$

- $\mathbf{p}(\mathbf{a})$ orthogonal to plane of $\mathbf{a}$, points RH direction
- $\mathbf{p}(\alpha\mathbf{a}) = \mathbf{p}(\mathbf{a}), \quad \alpha \neq 0$, magnitude of $\mathbf{a}$ has no effect
- $|\mathbf{p}(\mathbf{a})| = 2e/(e^2+1)$, $e =$ ellipticity (axial ratio)
- $\mathbf{p}(\mathbf{a})$ gives info on ellipticity, plane, and sense of rotation of $\mathbf{a}$
- $\mathbf{p}(\mathbf{a})$ does not give info on magnitude or orientation of ellipse on its plane

Vector Bases

- Three complex vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ form a basis if $\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3 \neq 0$
- Gibbs' identity by expanding $(\mathbf{a}_1 \times \mathbf{a}_2) \times (\mathbf{a}_3 \times \mathbf{b})$ in two ways:

  $$(\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3)\mathbf{b} = \mathbf{a}_1(\mathbf{a}_2 \times \mathbf{a}_3 \cdot \mathbf{b}) + \mathbf{a}_2(\mathbf{a}_3 \times \mathbf{a}_1 \cdot \mathbf{b}) + \mathbf{a}_3(\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{b})$$

- Define reciprocal basis

  $$\mathbf{a}'_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3}, \quad \mathbf{a}'_2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3}, \quad \mathbf{a}'_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3}$$

- Expansion for any vector $\mathbf{b}$

  $$\mathbf{b} = \sum_{i=1}^{3} \mathbf{a}_i(\mathbf{a}'_i \cdot \mathbf{b}) = \sum_{i=1}^{3} \mathbf{a}'_i(\mathbf{a}_i \cdot \mathbf{b})$$

Dyadic algebra

- Josiah Willard Gibbs 1884: dyadic algebra
- Dyadic = linear mapping from vector to vector
- Example 1: projection on line parallel to unit vector **u**

$$\mathbf{b} = \mathbf{u}(\mathbf{u}\cdot\mathbf{a}) = (\mathbf{u}\mathbf{u})\cdot\mathbf{a}$$

- **uu** = projection dyadic
- Example 2: projection on plane transverse to unit vector **u**

$$\mathbf{b} = \mathbf{a} - (\mathbf{u}\mathbf{u})\cdot\mathbf{a} = (\overline{\overline{I}} - \mathbf{u}\mathbf{u})\cdot\mathbf{a}$$

- $\overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{u}\mathbf{u}$ projection dyadic ($\overline{\overline{I}}$ unit dyadic, mapping to oneself)
- **uu** axial unit dyadic, $\overline{\overline{I}}_t$ transverse (planar) unit dyadic

Dyadic polynomial

- Dyad = dyadic product of two vectors $\mathbf{ab} \neq \mathbf{ba}$
- Dyadic = polynomial of dyads

$$\overline{\overline{A}} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \cdots + \mathbf{a}_N\mathbf{b}_N = \sum_{i=1}^{N} \mathbf{a}_i\mathbf{b}_i$$

- Dyadic as a mapping

$$\overline{\overline{A}} \cdot \mathbf{c} = (\sum_{i=1}^{N} \mathbf{a}_i\mathbf{b}_i) \cdot \mathbf{c} = \sum_{i=1}^{N} \mathbf{a}_i(\mathbf{b}_i \cdot \mathbf{c}) = \mathbf{a}_1(\mathbf{b}_1 \cdot \mathbf{c}) + \cdots + \mathbf{a}_N(\mathbf{b}_N \cdot \mathbf{c})$$

$$\mathbf{c} \cdot \overline{\overline{A}} = \mathbf{c} \cdot (\sum_{i=1}^{N} \mathbf{a}_i\mathbf{b}_i) = \sum_{i=1}^{N} (\mathbf{c} \cdot \mathbf{a}_i)\mathbf{b}_i = (\mathbf{c} \cdot \mathbf{a}_1)\mathbf{b}_1 + \cdots + (\mathbf{c} \cdot \mathbf{a}_N)\mathbf{b}_N$$

$$\mathbf{c} \cdot \overline{\overline{A}} = \overline{\overline{A}}^T \cdot \mathbf{c}, \quad \text{transposed dyadic} \quad \overline{\overline{A}}^T = \sum_{i=1}^{N} \mathbf{b}_i\mathbf{a}_i$$

Dyadic expressions

- There is no unique expression for dyadics. Two expressions represent same dyadic if they map all vectors in the same way:

$$\overline{\overline{A}}_1 \cdot \mathbf{c} = \overline{\overline{A}}_2 \cdot \mathbf{c}, \quad \text{for all } \mathbf{c}, \quad \Rightarrow \quad \overline{\overline{A}}_1 = \overline{\overline{A}}_2$$

- any dyadic can be expressed as a sum of three dyads

- Example: take a vector ONB $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$

$$\overline{\overline{A}} = \sum_{i=1}^{N} \mathbf{a}_i \mathbf{b}_i = \sum_{i=1}^{N} (\mathbf{u}_1 \mathbf{u}_1 \cdot \mathbf{a}_i + \mathbf{u}_2 \mathbf{u}_2 \cdot \mathbf{a}_i + \mathbf{u}_3 \mathbf{u}_3 \cdot \mathbf{a}_i) \mathbf{b}_i$$

$$= \mathbf{u}_1 \sum_{i=1}^{N} (\mathbf{u}_1 \cdot \mathbf{a}_i) \mathbf{b}_i + \mathbf{u}_2 \sum_{i=1}^{N} (\mathbf{u}_2 \cdot \mathbf{a}_i) \mathbf{b}_i + \mathbf{u}_3 \sum_{i=1}^{N} (\mathbf{u}_3 \cdot \mathbf{a}_i) \mathbf{b}_i$$

$$= \mathbf{u}_1 \mathbf{c}_1 + \mathbf{u}_2 \mathbf{c}_2 + \mathbf{u}_3 \mathbf{c}_3$$

- $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ arbitrary ON base $\Rightarrow$ infinite number of representations

Dyadic classification

- Any dyadic can be expressed as $\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \mathbf{a}_3\mathbf{b}_3$
- Planar dyadic can be expressed as sum of two dyads $\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2$
- Linear dyadic can be expressed as a single dyad $\mathbf{a}_1\mathbf{b}_1$
- Dyadic which cannot be expressed as a planar dyadic is complete
- Complete dyadic maps volumes to volumes
- Planar dyadic maps volumes to plane
- Linear dyadic maps volumes to line
- (Note: complex vectors $\Rightarrow$ planes and lines in complex space)
- Inverse mapping exists only for complete dyadics

Symmetric dyadics

- Symmetric dyadic $\overline{\overline{A}} = \overline{\overline{A}}^T = (1/2)(\overline{\overline{A}} + \overline{\overline{A}}^T)$

$$\sum_{i=1}^{3} \mathbf{a}_i\mathbf{b}_i = \frac{1}{2}\sum_{i=1}^{3}(\mathbf{a}_i\mathbf{b}_i+\mathbf{b}_i\mathbf{a}_i) = \frac{1}{4}\sum_{i=1}^{3}[(\mathbf{a}_i+\mathbf{b}_i)(\mathbf{a}_i+\mathbf{b}_i)-(\mathbf{a}_i-\mathbf{b}_i)(\mathbf{a}_i-\mathbf{b}_i)]$$

- Symmetric dyadic can be expressed as sum of symmetric dyads $\sum \mathbf{c}_i\mathbf{c}_i$ but not necessarily in three terms

- Unit dyadic symmetric $\overline{\overline{I}} = \mathbf{u}_1\mathbf{u}_1 + \mathbf{u}_2\mathbf{u}_2 + \mathbf{u}_3\mathbf{u}_3$ independent of ONB

- Examples: $\overline{\overline{I}} = \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y + \mathbf{u}_z\mathbf{u}_z = \mathbf{u}_r\mathbf{u}_r + \mathbf{u}_\theta\mathbf{u}_\theta + \mathbf{u}_\varphi\mathbf{u}_\varphi$

$$\overline{\overline{I}} \cdot \mathbf{a} = (\sum_{i=1}^{3} \mathbf{u}_i\mathbf{u}_i) \cdot \mathbf{a} = \sum_{i=1}^{3} \mathbf{u}_i(\mathbf{u}_i \cdot \mathbf{a})$$

Antisymmetric dyadics

- Antisymmetric dyadic $\overline{\overline{A}} = -\overline{\overline{A}}^T = (1/2)(\overline{\overline{A}} - \overline{\overline{A}}^T)$ operates through a vector $\mathbf{d}(\overline{\overline{A}})$

$$\overline{\overline{A}}\cdot\mathbf{c} = \sum_{i=1}^{3}\mathbf{a}_i(\mathbf{b}_i\cdot\mathbf{c}) = \frac{1}{2}\sum_{i=1}^{3}[\mathbf{a}_i(\mathbf{b}_i\cdot\mathbf{c}) - \mathbf{b}_i(\mathbf{a}_i\cdot\mathbf{c})] = \frac{1}{2}\sum_{i=1}^{3}(\mathbf{b}_i\times\mathbf{a}_i)\times\mathbf{c} = \mathbf{d}(\overline{\overline{A}})\times\mathbf{c}$$

$$\mathbf{d}(\overline{\overline{A}}) = \frac{1}{2}\sum_{i=1}^{3}\mathbf{b}_i\times\mathbf{a}_i = \frac{1}{2}(\mathbf{b}_1\times\mathbf{a}_1 + \mathbf{b}_2\times\mathbf{a}_2 + \mathbf{b}_3\times\mathbf{a}_3)$$

- Denoting $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times(\overline{\overline{I}}\cdot\mathbf{b}) = (\mathbf{a}\times\overline{\overline{I}})\cdot\mathbf{b}$,

$$\overline{\overline{A}} = \sum_{i=1}^{3}\mathbf{a}_i\mathbf{b}_i = \frac{1}{2}\sum_{i=1}^{3}(\mathbf{b}_i\times\mathbf{a}_i)\times\overline{\overline{I}} = \mathbf{d}(\overline{\overline{A}})\times\overline{\overline{I}}$$

- Any dyadic of the form $\mathbf{a}\times\overline{\overline{I}} = \overline{\overline{I}}\times\mathbf{a}$ is antisymmetric

Products of dyadics and vectors

- Dot products $\overline{\overline{A}} \cdot \mathbf{c}$ and $\mathbf{c} \cdot \overline{\overline{A}}$ give vectors

$$\overline{\overline{A}} \cdot \mathbf{c} = (\sum \mathbf{a}_i \mathbf{b}_i) \cdot \mathbf{c} = \sum \mathbf{a}_i (\mathbf{b}_i \cdot \mathbf{c})$$

$$\mathbf{c} \cdot \overline{\overline{A}} = \mathbf{c} \cdot (\sum \mathbf{a}_i \mathbf{b}_i) = \sum (\mathbf{c} \cdot \mathbf{a}_i) \mathbf{b}_i$$

- Example: antisymmetric dyadic $\overline{\overline{A}} = \mathbf{a} \times \overline{\overline{I}} = \overline{\overline{I}} \times \mathbf{a}$

$$\overline{\overline{A}} \cdot \mathbf{c} = (\mathbf{a} \times \overline{\overline{I}}) \cdot \mathbf{c} = \mathbf{a} \times \mathbf{c}, \qquad \mathbf{c} \cdot \overline{\overline{A}} = \mathbf{c} \cdot (\mathbf{a} \times \overline{\overline{I}}) = \mathbf{c} \times \mathbf{a}$$

- Cross products $\overline{\overline{A}} \times \mathbf{c}$ and $\mathbf{c} \times \overline{\overline{A}}$ give dyadics

$$\overline{\overline{A}} \times \mathbf{c} = (\sum \mathbf{a}_i \mathbf{b}_i) \times \mathbf{c} = \sum \mathbf{a}_i (\mathbf{b}_i \times \mathbf{c})$$

$$\mathbf{c} \times \overline{\overline{A}} = \mathbf{c} \times (\sum \mathbf{a}_i \mathbf{b}_i) = \sum (\mathbf{c} \times \mathbf{a}_i) \mathbf{b}_i$$

- Note: $\mathbf{a} \cdot (\mathbf{b} \times \overline{\overline{A}}) = (\mathbf{a} \times \mathbf{b}) \cdot \overline{\overline{A}}$ but $\neq \mathbf{a} \times (\mathbf{b} \cdot \overline{\overline{A}})$!

Dot-product of dyadics

- Dot product $\overline{\overline{A}} \cdot \overline{\overline{B}}$ gives a dyadic

$$\overline{\overline{A}} \cdot \overline{\overline{B}} = (\sum_i \mathbf{a}_i \mathbf{b}_i) \cdot (\sum_j \mathbf{c}_j \mathbf{d}_j) = \sum_{i,j} (\mathbf{b}_i \cdot \mathbf{c}_j) \mathbf{a}_i \mathbf{d}_j$$

- Dot product is associative but not commutative like matrix product

$$\overline{\overline{A}} \cdot (\overline{\overline{B}} \cdot \overline{\overline{C}}) = (\overline{\overline{A}} \cdot \overline{\overline{B}}) \cdot \overline{\overline{C}}, \qquad \overline{\overline{A}} \cdot \overline{\overline{B}} \neq \overline{\overline{B}} \cdot \overline{\overline{A}} \text{ (in general)}$$

- Powers of dyadics

$$\overline{\overline{A}}^2 = \overline{\overline{A}} \cdot \overline{\overline{A}}, \quad \overline{\overline{A}}^n = \overline{\overline{A}} \cdot \overline{\overline{A}}^{n-1} = \overline{\overline{A}}^{n-1} \cdot \overline{\overline{A}}, \quad \overline{\overline{A}}^0 = \overline{\overline{I}}$$

- Inverse of a dyadic possible for complete dyadics only

$$\overline{\overline{A}} \cdot \mathbf{a} = \mathbf{b}, \quad \Rightarrow \quad \mathbf{a} = \overline{\overline{A}}^{-1} \cdot \mathbf{b}$$

$$(\overline{\overline{A}} \cdot \overline{\overline{B}})^{-1} = \overline{\overline{B}}^{-1} \cdot \overline{\overline{A}}^{-1}$$

Double-cross product of dyadics

- Double-cross product $\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}$ gives a dyadic

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}} = (\sum_i \mathbf{a}_i\mathbf{b}_i){}^{\times}_{\times}(\sum_j \mathbf{c}_j\mathbf{d}_j) = \sum_{i,j}(\mathbf{a}_i \times \mathbf{c}_j)(\mathbf{b}_i \times \mathbf{d}_j)$$

- Double-cross product is commutative but not associative

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}} = \overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{A}} \qquad \overline{\overline{A}}{}^{\times}_{\times}(\overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{C}}) \neq (\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}){}^{\times}_{\times}\overline{\overline{C}} \text{ (in general)}$$

- Double-cross square

$$\overline{\overline{A}}^{(2)} = \frac{1}{2}\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}, \qquad \overline{\overline{I}}^{(2)} = \overline{\overline{I}}$$

- Inverse of a dyadic can be expressed as

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^{(2)T}}{\det\overline{\overline{A}}}$$

Double-dot product of dyadics

- Double-dot product $\overline{\overline{A}}:\overline{\overline{B}}$ gives a scalar

$$\overline{\overline{A}}:\overline{\overline{B}} = (\sum_i \mathbf{a}_i\mathbf{b}_i):(\sum_j \mathbf{c}_j\mathbf{d}_j) = \sum_{i,j}(\mathbf{a}_i\cdot\mathbf{c}_j)(\mathbf{b}_i\cdot\mathbf{d}_j)$$

$$\overline{\overline{A}}:\overline{\overline{B}} = \overline{\overline{B}}:\overline{\overline{A}} = \overline{\overline{A}}^T:\overline{\overline{B}}^T, \quad \overline{\overline{A}}:\overline{\overline{B}}^T = \overline{\overline{B}}:\overline{\overline{A}}^T$$

- If $\overline{\overline{A}}$ antisymmetric and $\overline{\overline{B}}$ symmetric, $\overline{\overline{A}}:\overline{\overline{B}} = 0$

$$\mathbf{ab}:\overline{\overline{A}} = \mathbf{a}\cdot\overline{\overline{A}}\cdot\mathbf{b}, \quad \mathbf{ab}:\overline{\overline{I}} = \mathbf{a}\cdot\mathbf{b}, \quad \overline{\overline{A}}:\overline{\overline{B}} = (\overline{\overline{A}}\cdot\overline{\overline{B}}^T):\overline{\overline{I}}$$

$$\overline{\overline{A}}:\overline{\overline{I}} = (\sum \mathbf{a}_i\mathbf{b}_i):\overline{\overline{I}} = \sum \mathbf{a}_i\cdot\mathbf{b}_i = \mathrm{tr}\overline{\overline{A}} \quad \mathrm{tr}\overline{\overline{I}} = 3, \quad \text{trace}$$

$$\overline{\overline{A}}^{(2)}:\overline{\overline{I}} = \frac{1}{2}(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}):\overline{\overline{I}} = \mathrm{spm}\overline{\overline{A}} \quad \text{'sum of principal minors'}$$

$$\det\overline{\overline{A}} = \frac{1}{6}(\overline{\overline{A}}{}^\times_\times\overline{\overline{A}}):\overline{\overline{A}} = \frac{1}{3}\overline{\overline{A}}^{(2)}:\overline{\overline{A}} \quad \text{determinant}$$

Dyadic identities

- Dyadic identities are needed in dyadic analysis. They can be formed (1) through vector expansions or (2) from other identities. Example of (1) $\overline{\overline{A}}{}^{\times}_{\times}(\overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{C}}) = ?$ Procedure:

- Expand the dyadics. For brevity omit indices and sum signs, $\overline{\overline{A}} \rightarrow$ **ab**, $\overline{\overline{B}} \rightarrow$ **cd**, $\overline{\overline{C}} \rightarrow$ **ef**. Apply vector identities.

$$(\mathbf{ab})^{\times}_{\times}[(\mathbf{cd})^{\times}_{\times}(\mathbf{ef})] = [\mathbf{a}\times(\mathbf{c}\times\mathbf{e})][\mathbf{b}\times(\mathbf{d}\times\mathbf{f})] = [\mathbf{c}(\mathbf{a}\cdot\mathbf{e})-\mathbf{e}(\mathbf{a}\cdot\mathbf{c})][\mathbf{d}(\mathbf{b}\cdot\mathbf{f})-\mathbf{f}(\mathbf{b}\cdot\mathbf{d})]$$

- Write result in terms of dyadic products **ab**, **cd** and **ef**

$$= (\mathbf{ab} : \mathbf{ef})\mathbf{cd} - (\mathbf{ef}) \cdot (\mathbf{ba}) \cdot (\mathbf{cd}) - (\mathbf{cd}) \cdot (\mathbf{ba}) \cdot (\mathbf{ef}) + (\mathbf{ab} : \mathbf{cd})\mathbf{ef}$$

- Replace $\mathbf{ab} \rightarrow \overline{\overline{A}}$, $\mathbf{cd} \rightarrow \overline{\overline{B}}$, $\mathbf{ef} \rightarrow \overline{\overline{C}}$, results identity

$$\overline{\overline{A}}{}^{\times}_{\times}(\overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{C}}) = (\overline{\overline{A}} : \overline{\overline{C}})\overline{\overline{B}} - \overline{\overline{C}} \cdot \overline{\overline{A}}^T \cdot \overline{\overline{B}} - \overline{\overline{B}} \cdot \overline{\overline{A}}^T \cdot \overline{\overline{C}} + (\overline{\overline{A}} : \overline{\overline{B}})\overline{\overline{C}}$$

More dyadic identities

- Another dyadic identity $(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}):\overline{\overline{I}}=?$ similarly

$$[(\mathbf{ab})^\times_\times(\mathbf{cd})]:\overline{\overline{I}}=[(\mathbf{a}\times\mathbf{c})(\mathbf{b}\times\mathbf{d})]:\overline{\overline{I}}=(\mathbf{a}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})$$

$$=(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{b})=(\mathbf{ab}:\overline{\overline{I}})(\mathbf{cd}:\overline{\overline{I}})-(\mathbf{ab}):(\mathbf{cd})^T$$

- Resulting identity $(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}):\overline{\overline{I}}=(\overline{\overline{A}}:\overline{\overline{I}})(\overline{\overline{B}}:\overline{\overline{I}})-\overline{\overline{A}}:\overline{\overline{B}}^T$ valid for any dyadics $\overline{\overline{A}}$ and $\overline{\overline{B}}$. A new identity obtained by writing [note that $(\overline{\overline{A}}{}^\times_\times\overline{\overline{B}}):\overline{\overline{C}}=\overline{\overline{A}}:(\overline{\overline{B}}{}^\times_\times\overline{\overline{C}})$]

$$\overline{\overline{A}}:[\overline{\overline{B}}{}^\times_\times\overline{\overline{I}}-(\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{I}}+\overline{\overline{B}}^T]=0$$

- $\overline{\overline{A}}:\overline{\overline{C}}=0$ for all dyadics $\overline{\overline{A}}$, implies $\overline{\overline{C}}=0$.

  New identity: $\overline{\overline{B}}{}^\times_\times\overline{\overline{I}}=(\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{I}}-\overline{\overline{B}}^T$

The inverse dyadic 1

- The general dyadic involves two vector bases $\{\mathbf{a}_i\}, \{\mathbf{b}_i\}$:

$$\overline{\overline{A}} = \sum_{i=1}^{3} \mathbf{a}_i \mathbf{b}_i = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \mathbf{a}_3\mathbf{b}_3$$

- The determinant and double-cross square become

$$\det\overline{\overline{A}} = \frac{1}{6}\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}} : \overline{\overline{A}} = \mathbf{a}_1\mathbf{b}_1{}^{\times}_{\times}\mathbf{a}_2\mathbf{b}_2 : \mathbf{a}_3\mathbf{b}_3 = (\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3)(\mathbf{b}_1 \times \mathbf{b}_2 \cdot \mathbf{b}_3)$$

$$\overline{\overline{A}}^{(2)} = \frac{1}{2}\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}} = \mathbf{a}_2\mathbf{b}_2{}^{\times}_{\times}\mathbf{a}_3\mathbf{b}_3 + \mathbf{a}_3\mathbf{b}_3{}^{\times}_{\times}\mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_1\mathbf{b}_1{}^{\times}_{\times}\mathbf{a}_2\mathbf{b}_2$$

$$= (\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3)(\mathbf{b}_1 \times \mathbf{b}_2 \cdot \mathbf{b}_3)[\mathbf{a}'_1\mathbf{b}'_1 + \mathbf{a}'_2\mathbf{b}'_2 + \mathbf{a}'_3\mathbf{b}'_3] = \det\overline{\overline{A}} \sum_i \mathbf{a}'_i\mathbf{b}'_i$$

- Here $\{\mathbf{a}'_i\}$ and $\{\mathbf{b}'_i\}$ are bases reciprocal to $\{\mathbf{a}_i\}$ and $\{\mathbf{b}_i\}$

The inverse dyadic 2

- Orthogonality of reciprocal bases $\mathbf{b}_i \cdot \mathbf{b}'_j = \delta_{ij}$ and Gibbs' identity $\mathbf{c} = \sum \mathbf{a}_i \mathbf{a}'_i \cdot \mathbf{c}$ give an identity

$$\overline{\overline{A}} \cdot \overline{\overline{A}}^{(2)T} = \det\overline{\overline{A}} \sum_i \mathbf{a}_i \mathbf{b}_i \cdot \sum_j \mathbf{b}'_j \mathbf{a}'_j = \det\overline{\overline{A}} \sum_i \mathbf{a}_i \mathbf{a}'_i = \det\overline{\overline{A}}\ \overline{\overline{I}}$$

$$\overline{\overline{A}} \cdot \overline{\overline{A}}^{(2)T} = \overline{\overline{A}}^{(2)T} \cdot \overline{\overline{A}} = \det\overline{\overline{A}}\ \overline{\overline{I}}$$

- For complete dyadic satisfying $\det\overline{\overline{A}} \neq 0$ the inverse becomes

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^{(2)T}}{\det\overline{\overline{A}}} = \frac{\frac{1}{2}\overline{\overline{A}}^T {}^\times_\times \overline{\overline{A}}^T}{\frac{1}{6}\overline{\overline{A}} {}^\times_\times \overline{\overline{A}} : \overline{\overline{A}}}$$

- For planar dyadics $\det\overline{\overline{A}} = 0$ no inverse exists

Example of an inverse dyadic

- Find the inverse of $\overline{\overline{B}} = \alpha\overline{\overline{I}} + \mathbf{a}\times\overline{\overline{I}}$

$$\overline{\overline{B}}^{(2)} = \frac{1}{2}[(\alpha\overline{\overline{I}})^\times_\times(\alpha\overline{\overline{I}})+2(\alpha\overline{\overline{I}})^\times_\times(\mathbf{a}\times\overline{\overline{I}})+(\mathbf{a}\times\overline{\overline{I}})^\times_\times(\mathbf{a}\times\overline{\overline{I}})] = \alpha^2\overline{\overline{I}}+\alpha(\mathbf{a}\times\overline{\overline{I}})+\mathbf{aa}$$

- To evaluate we apply the properties

$$\overline{\overline{I}}^{(2)} = \overline{\overline{I}}, \quad (\mathbf{a}\times\overline{\overline{I}})^\times_\times\overline{\overline{I}} = -(\mathbf{a}\times\overline{\overline{I}})^T = \mathbf{a}\times\overline{\overline{I}}, \quad (\mathbf{a}\times\overline{\overline{I}})^\times_\times(\mathbf{a}\times\overline{\overline{I}}) = 2\mathbf{aa}$$

$$(\mathbf{a}\times\overline{\overline{I}}):(\mathbf{a}\times\overline{\overline{I}}) = 2\mathbf{a}\cdot\mathbf{a}, \quad \overline{\overline{I}}:\overline{\overline{I}} = 3, \quad \overline{\overline{I}}:(\mathbf{a}\times\overline{\overline{I}}) = 0$$

$$\mathrm{det}\overline{\overline{B}} = \frac{1}{3}\overline{\overline{B}}^{(2)}:\overline{\overline{B}} = \alpha(\alpha^2+\mathbf{a}\cdot\mathbf{a})$$

- The result becomes

$$\overline{\overline{B}}^{-1} = \frac{1}{\alpha(\alpha^2+\mathbf{a}\cdot\mathbf{a})}[\alpha^2\overline{\overline{I}} - \alpha(\mathbf{a}\times\overline{\overline{I}}) + \mathbf{aa}]$$

Problems

1.1 Derive the dyadic identity

$$(\mathbf{a} \times \overline{\overline{A}})^{(2)} = \mathbf{aa} \cdot \overline{\overline{A}}^{(2)}$$

by starting from the expression $(\mathbf{a} \times \mathbf{bc}) {}^{\times}_{\times} (\mathbf{a} \times \mathbf{de})$.

1.2 Given a symmetric dyadic $\overline{\overline{S}}$ and a vector $\mathbf{a}$ show that one can find a vector $\mathbf{b}$ satisfying

$$\overline{\overline{S}} \times \mathbf{a} + \mathbf{a} \times \overline{\overline{S}} = \mathbf{b} \times \overline{\overline{I}},$$

i.e., the dyadic on the left is antisymmetric. Find the vector $\mathbf{b}$.

S-96.510 Advanced Field Theory
2. Dyadic Algebra

I.V.Lindell

Dyadics and matrices 1

- Dyadics can be expanded in any ONB $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ as

$$\overline{\overline{A}} = \overline{\overline{I}} \cdot \overline{\overline{A}} \cdot \overline{\overline{I}} = \sum \mathbf{u}_i \mathbf{u}_i \cdot \overline{\overline{A}} \cdot \sum \mathbf{u}_j \mathbf{u}_j = \sum \sum A_{ij} \mathbf{u}_i \mathbf{u}_j,$$

- Matrix elements $A_{ij} = \overline{\overline{A}} : \mathbf{u}_i \mathbf{u}_j$ depend on chosen vector basis. Some quantities (invariants) are independent of the basis:

$$\mathrm{tr}\overline{\overline{A}} = \overline{\overline{A}} : \overline{\overline{I}} = \sum_i \overline{\overline{A}} : \mathbf{u}_i \mathbf{u}_i = \sum_i A_{ii}$$

$$\mathrm{spm}\overline{\overline{A}} = \overline{\overline{A}}^{(2)} : \overline{\overline{I}} = \frac{1}{2} \sum_{i,j,k,\ell} A_{ij} A_{k\ell} (\mathbf{u}_i \times \mathbf{u}_k) \cdot (\mathbf{u}_j \times \mathbf{u}_\ell)$$

$$= \frac{1}{2} \sum_{i,j,k,\ell} A_{ij} A_{k\ell} (\delta_{ij}\delta_{k\ell} - \delta_{i\ell}\delta_{jk}) = \frac{1}{2} \sum_{i,k} (A_{ii} A_{kk} - A_{ik} A_{ki})$$

$$= A_{11} A_{22} - A_{12} A_{21} + \cdots = \frac{1}{2} [(\overline{\overline{A}} : \overline{\overline{I}})^2 - \overline{\overline{A}} : \overline{\overline{A}}^T]$$

Dyadics and matrices 2

- Third scalar invariant = determinant

$$\mathrm{det}\overline{\overline{A}} = \frac{1}{6}\sum A_{ij}A_{k\ell}A_{mn}(\mathbf{u}_i \times \mathbf{u}_k \cdot \mathbf{u}_m)(\mathbf{u}_j \times \mathbf{u}_\ell \cdot \mathbf{u}_n)$$

$$= \frac{1}{6}\sum A_{ij}A_{k\ell}A_{mn}\epsilon_{ikm}\epsilon_{j\ell n} = \mathrm{det}(A_{ij})$$

- Dot-product of dyadics corresponds to the matrix product:

$$\overline{\overline{A}} \cdot \overline{\overline{B}} = \sum A_{ij}B_{k\ell}\mathbf{u}_i\mathbf{u}_j \cdot \mathbf{u}_k\mathbf{u}_\ell = \sum_{i,\ell}\mathbf{u}_i\mathbf{u}_\ell\sum_j A_{ij}B_{j\ell}$$

- Double-cross product = 'mixed subdeterminant' in matrix algebra

$$\overline{\overline{A}}{}^\times_\times\overline{\overline{B}} = \sum_{i,j,k,\ell} A_{ij}B_{k\ell}(\mathbf{u}_i \times \mathbf{u}_k)(\mathbf{u}_j \times \mathbf{u}_\ell)$$

$$= \mathbf{u}_1\mathbf{u}_1(A_{22}B_{33} + A_{33}B_{22} - A_{23}B_{32} - A_{32}B_{23}) + \cdots$$

Important identity

- A useful identity valid for all $\overline{\overline{A}}, \mathbf{a}, \mathbf{b}$:

$$\overline{\overline{A}}^{(2)} \cdot (\mathbf{a} \times \mathbf{b}) = (\overline{\overline{A}} \cdot \mathbf{a}) \times (\overline{\overline{A}} \cdot \mathbf{b})$$

$$\text{or} \quad [\overline{\overline{A}}^{(2)} \times \mathbf{a} - (\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{A}}] \cdot \mathbf{b} = 0 \quad \text{for all } \mathbf{b}$$

  can be generalized to

$$\overline{\overline{A}}^{(2)} \times \mathbf{a} = (\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{A}}$$

- Further generalization: substitute $\overline{\overline{A}} \to \overline{\overline{A}} + \overline{\overline{B}}$ and cancel terms transforms the quadratic identity to a bilinear one:

$$(\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{B}}) \times \mathbf{a} = (\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{B}} + (\overline{\overline{B}} \cdot \mathbf{a}) \times \overline{\overline{A}}$$

- Proof of the identity quadratic in $\overline{\overline{A}}$ through identity linear in $\overline{\overline{A}}, \overline{\overline{B}}$

$$(\mathbf{bc} {}^{\times}_{\times} \mathbf{de}) \times \mathbf{a} = (\mathbf{b} \times \mathbf{d})[\mathbf{e}(\mathbf{c} \cdot \mathbf{a}) - \mathbf{c}(\mathbf{e} \cdot \mathbf{a})] = (\mathbf{bc} \cdot \mathbf{a}) \times (\mathbf{de}) + (\mathbf{de} \cdot \mathbf{a}) \times (\mathbf{bc})$$

Classification of dyadics

- $\overline{\overline{A}}$ is a **complete** dyadic iff $\det\overline{\overline{A}} \neq 0, \quad \Rightarrow \overline{\overline{A}}^{-1}$ exists
- $\overline{\overline{A}}$ is a **planar** dyadic iff $\det\overline{\overline{A}} = 0$ or exists $\mathbf{a} \neq 0$, $\overline{\overline{A}} \cdot \mathbf{a} = 0$
- $\overline{\overline{A}}$ is a **linear** dyadic iff $\overline{\overline{A}}^{(2)} = 0$ or exists $\mathbf{a} \neq 0$, $\overline{\overline{A}} \times \mathbf{a} = 0$

$$\text{identity} \qquad \overline{\overline{A}}^{(2)} \times \mathbf{a} = (\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{A}}$$

- conclusion: if $\overline{\overline{A}}$ planar, $\overline{\overline{A}}^{(2)}$ linear

$$\text{identity} \qquad (\overline{\overline{A}}^{(2)})^{(2)} = \overline{\overline{A}}\det\overline{\overline{A}} = \overline{\overline{A}}\sqrt{\det\overline{\overline{A}}^{(2)}}$$

- conclusion: if $\overline{\overline{A}}^{(2)}$ planar it is also linear

Eigenvalue problems

- Eigenvalues λ and (right) eigenvectors $\mathbf{a}$ of a dyadic $\overline{\overline{A}}$ satisfy

$$\overline{\overline{A}}\cdot\mathbf{a}=\lambda\mathbf{a},\quad \mathbf{a}\neq 0,\quad \Rightarrow \quad (\overline{\overline{A}}-\lambda\overline{\overline{I}})\cdot\mathbf{a}=0$$

- $\overline{\overline{A}}-\lambda\overline{\overline{I}}$ planar $\Rightarrow$ $\det(\overline{\overline{A}}-\lambda\overline{\overline{I}})=0$ 3rd degree equation for λ

$$\frac{1}{6}(\overline{\overline{A}}-\lambda\overline{\overline{I}})^{\times}_{\times}(\overline{\overline{A}}-\lambda\overline{\overline{I}}):(\overline{\overline{A}}-\lambda\overline{\overline{I}})=\det\overline{\overline{A}}-\lambda\mathrm{spm}\overline{\overline{A}}+\lambda^2\mathrm{tr}\overline{\overline{A}}-\lambda^3=0$$

- Three roots satisfy the conditions

$$\mathrm{tr}\overline{\overline{A}}=\lambda_1+\lambda_2+\lambda_3,\quad \mathrm{spm}\overline{\overline{A}}=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1,\quad \det\overline{\overline{A}}=\lambda_1\lambda_2\lambda_3$$

- General case: three different eigenvalues $\lambda_1,\lambda_2,\lambda_3$
- Two eigenvalues same if $\overline{\overline{A}}$ of the form $\alpha\overline{\overline{I}}+\mathbf{bc}$ (generalized uniaxial dyadic)
- Three eigenvalues same if $\overline{\overline{A}}$ of the form $\alpha\overline{\overline{I}}$ (isotropic dyadic)

Finding the eigenvectors

- Single eigenvalues $\lambda_i \Rightarrow$ eigenvectors $\mathbf{a}_i$ from

$$(\overline{\overline{A}} - \lambda_i \overline{\overline{I}}) \cdot \mathbf{a}_i = 0, \quad \Rightarrow \quad (\overline{\overline{A}} - \lambda_i \overline{\overline{I}})^{(2)} \times \mathbf{a}_i = 0$$

- $(\overline{\overline{A}} - \lambda_i \overline{\overline{I}})^{(2)}$ is a linear dyadic of the form $\mathbf{b}_i \mathbf{a}_i$ ($\mathbf{b}_i$ left eigenvector)

$$\mathbf{a}_i = \mathbf{c} \cdot (\overline{\overline{A}} - \lambda_i \overline{\overline{I}})^{(2)}, \quad \text{when } \neq 0$$

- For generalized uniaxial dyadic $\overline{\overline{A}} = \alpha \overline{\overline{I}} + \mathbf{bc}$ double eigenvalue $\lambda_{2,3} = \alpha$, $\Rightarrow \overline{\overline{A}} - \alpha \overline{\overline{I}} = \mathbf{bc}$ linear dyadic, $(\overline{\overline{A}} - \alpha \overline{\overline{I}})^{(2)} = 0$ any vectors $\mathbf{a}_{2,3}$ satisfying $\mathbf{c} \cdot \mathbf{a}_{2,3} = 0$ are eigenvectors

- For isotropic dyadic $\overline{\overline{A}} = \alpha \overline{\overline{I}}$ triple eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = \alpha$, all vectors are eigenvectors

Uniaxial dyadics 1

- Uniaxial dyadics (symmetric) are encountered in medium equations and interface conditions. Axial real unit vector $\mathbf{u}$, $\overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{uu}$ (Notation different from that in the book!)

$$\overline{\overline{U}}(\alpha,\beta) = \alpha\overline{\overline{I}}_t + \beta\mathbf{uu}, \quad [\text{book}: \overline{\overline{D}}(\alpha,\beta) = \alpha\overline{\overline{I}} + \beta\mathbf{uu} = \overline{\overline{U}}(\alpha, \alpha+\beta)]$$

$$\text{Matrix notation}: \quad \overline{\overline{U}}(\alpha,\beta) \rightarrow \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

- Uniaxial dyadics form a two-dimensional linear space:

$$\overline{\overline{U}}(\alpha_1,\beta_1) + \overline{\overline{U}}(\alpha_2,\beta_2) = \overline{\overline{U}}(\alpha_1+\alpha_2, \beta_1+\beta_2),$$

$$\tau\overline{\overline{U}}(\alpha,\beta) = \overline{\overline{U}}(\tau\alpha, \tau\beta)$$

$$\overline{\overline{U}}(1,1) = \overline{\overline{I}}, \quad \overline{\overline{U}}(1,0) = \overline{\overline{I}}_t, \quad \overline{\overline{U}}(0,1) = \mathbf{uu} \quad \text{special cases}$$

- Eigensolutions $\mathbf{a}_1 = \mathbf{u}$, $\lambda_1 = \beta$ and $\mathbf{a}_{2,3} \perp \mathbf{u}$, $\lambda_{2,3} = \alpha$

Uniaxial dyadics 2

- Properties of symmetric uniaxial dyadics

$$\overline{\overline{U}}(\alpha_1,\beta_1)\cdot\overline{\overline{U}}(\alpha_2,\beta_2)=\overline{\overline{U}}(\alpha_1\alpha_2,\beta_1\beta_2),$$

$$\overline{\overline{U}}^n(\alpha,\beta)=\overline{\overline{U}}(\alpha^n,\beta^n),\qquad \overline{\overline{U}}^{-1}(\alpha,\beta)=\overline{\overline{U}}(\alpha^{-1},\beta^{-1}),$$

$$\overline{\overline{U}}(\alpha_1,\beta_1){}^\times_\times\overline{\overline{U}}(\alpha_2,\beta_2)=\overline{\overline{U}}(\alpha_1\beta_2+\alpha_2\beta_1,2\alpha_1\alpha_2)\quad\Rightarrow\quad\overline{\overline{U}}^{(2)}(\alpha,\beta)=\overline{\overline{U}}(\alpha\beta,\alpha^2)$$

$$\overline{\overline{U}}(\alpha_1,\beta_1):\overline{\overline{U}}(\alpha_2,\beta_2)=2\alpha_1\alpha_2+\beta_1\beta_2,\qquad \mathrm{det}\overline{\overline{U}}(\alpha,\beta)=\alpha^2\beta$$

- Checking the inverse

$$\overline{\overline{U}}^{-1}(\alpha,\beta)=\frac{\overline{\overline{U}}^{(2)T}(\alpha,\beta)}{\mathrm{det}\overline{\overline{U}}(\alpha,\beta)}=\frac{\overline{\overline{U}}(\alpha\beta,\alpha^2)}{\alpha^2\beta}=\overline{\overline{U}}(\alpha^{-1},\beta^{-1})$$

- Generalized uniaxial dyadics $\alpha\overline{\overline{I}}+\mathbf{bc}$ obey more complicated rules

Reflection dyadic

- Reflection dyadic $\overline{\overline{C}}$ giving mirror image of vector $\mathbf{a}$ in the plane $\mathbf{u}\cdot\mathbf{r}=0$ is uniaxial:

$$\overline{\overline{C}}\cdot\mathbf{a}=\mathbf{a}-2\mathbf{u}(\mathbf{u}\cdot\mathbf{a})=(\overline{\overline{I}}-2\mathbf{uu})\cdot\mathbf{a},$$

$$\overline{\overline{C}}=\overline{\overline{I}}_t-\mathbf{uu}=\overline{\overline{U}}(1,-1),\quad\Rightarrow\quad\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

- Properties of the reflection dyadic

$$\mathrm{tr}\overline{\overline{C}}=1,\quad \mathrm{spm}\overline{\overline{C}}=-1,\quad \det\overline{\overline{C}}=-1,\quad \overline{\overline{C}}^{(2)}=-\overline{\overline{C}},\quad \overline{\overline{C}}^{-1}=\overline{\overline{C}}$$

- $\overline{\overline{C}}^2=\overline{\overline{I}},\quad\Rightarrow\quad\overline{\overline{C}}=\overline{\overline{I}}^{1/2}$
- Square root of a dyadic is not unique!
- Eigensolutions $\lambda_1=-1,\mathbf{a}_1=\mathbf{u},\ \ \lambda_{2,3}=1,\mathbf{a}_{2,3}\perp\mathbf{u}$

Gyrotropic dyadics 1

- Gyrotropic dyadic = uniaxial dyadic + co-axial antisymmetric dyadic (Notation in the book again obtained by substituting $\beta \to \alpha + \beta$)

$$\overline{\overline{G}}(\alpha, \beta, \gamma) = \alpha \overline{\overline{I}}_t + \beta \mathbf{uu} + \gamma \overline{\overline{J}}$$

$$\overline{\overline{J}} = \mathbf{u}\times\overline{\overline{I}}, \quad \overline{\overline{J}}^2 = (\mathbf{u}\times\overline{\overline{I}})\cdot(\mathbf{u}\times\overline{\overline{I}}) = \mathbf{u}\times(\mathbf{u}\times\overline{\overline{I}}) = -\overline{\overline{I}}_t, \quad \overline{\overline{J}}^3 = -\overline{\overline{J}}, \; \cdots$$

$$\overline{\overline{G}}(\alpha, \beta, \gamma) \to \begin{pmatrix} \alpha & \gamma & 0 \\ -\gamma & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}, \qquad \overline{\overline{G}}(\alpha, \beta, 0) = \overline{\overline{U}}(\alpha, \beta)$$

- Gyrotropic dyadics are encountered in magnetoplasmas and ferrites

$$\overline{\overline{\epsilon}}_r = \left(1 - \frac{\omega_g \omega_p^2}{\omega(\omega_g^2 - \omega^2)}\right) \overline{\overline{I}}_t + \left(1 - \frac{\omega_p^2}{\omega^2}\right) \mathbf{uu} + \frac{j\omega_g \omega_p^2}{\omega(\omega_g^2 - \omega^2)} \overline{\overline{J}}$$

$$\overline{\overline{\mu}}_r = \left(1 - \frac{\omega_o \omega_m}{\omega_o^2 - \omega^2}\right) \overline{\overline{I}}_t + \mathbf{uu} + j \frac{\omega \omega_m}{\omega_o^2 - \omega^2} \overline{\overline{J}}$$

Gyrotropic dyadics 2

- Two-dimensional (planar) part of gyrotropic dyadic can be expressed as
$$\overline{\overline{G}}_t(\alpha,\gamma) = \alpha\overline{\overline{I}}_t + \gamma\overline{\overline{J}} = Ge^{\overline{\overline{J}}\theta},$$
- $e^{\overline{\overline{J}}\theta}$ defined here as two-dimensional exponential function
$$e^{\overline{\overline{J}}\theta} = \overline{\overline{I}}_t + \theta\overline{\overline{J}} + \frac{\theta^2}{2!}\overline{\overline{J}}^2 + \frac{\theta^3}{3!}\overline{\overline{J}}^3 + \cdots = \cos\theta\overline{\overline{I}}_t + \sin\theta\overline{\overline{J}}$$
- Planar rotation defined through dyadic $\overline{\overline{R}}_t(\theta)$ as
$$\overline{\overline{R}}_t(\theta) = \cos\theta\ \overline{\overline{I}}_t + \sin\theta\overline{\overline{J}} = e^{\overline{\overline{J}}\theta}$$
$$\overline{\overline{G}}_t(\alpha,\gamma) = G\overline{\overline{R}}_t(\theta), \qquad G = \sqrt{\alpha^2+\gamma^2}, \quad \tan\theta = \gamma/\alpha$$
- Note the similarity to operating with complex numbers!

Gyrotropic dyadics 3

- Denoting $\cos\theta = \alpha/G$, $\sin\theta = \gamma/G$, $G = \sqrt{\alpha^2+\gamma^2}$

$$\overline{\overline{G}}(\alpha,\beta,\gamma) = \beta\mathbf{u}\mathbf{u} + Ge^{\overline{\overline{J}}\theta},$$

- gives the multiplication rule

$$\overline{\overline{G}}(\alpha_1,\beta_1,\gamma_1)\cdot\overline{\overline{G}}(\alpha_2,\beta_2,\gamma_2) = \beta_1\beta_2\mathbf{u}\mathbf{u} + G_1G_2e^{\overline{\overline{J}}(\theta_1+\theta_2)} =$$

$$\beta_1\beta_2\mathbf{u}\mathbf{u}+G_1G_2[(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2)\overline{\overline{I}}_t+(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)\overline{\overline{J}}]$$

$$= \overline{\overline{G}}(\alpha_1\alpha_2-\gamma_1\gamma_2, \beta_1\beta_2, \alpha_1\gamma_2+\alpha_2\gamma_1)$$

$$\overline{\overline{G}}^2(\alpha,\beta,\gamma) = \beta^2\mathbf{u}\mathbf{u} + G^2e^{\overline{\overline{J}}2\theta} = \overline{\overline{G}}(\alpha^2-\gamma^2, \beta^2, 2\alpha\gamma),$$

$$\overline{\overline{G}}^n(\alpha,\beta,\gamma) = \beta^n\mathbf{u}\mathbf{u} + G^ne^{\overline{\overline{J}}n\theta},$$

$$\overline{\overline{G}}^{-1}(\alpha,\beta,\gamma) = \beta^{-1}\mathbf{u}\mathbf{u} + G^{-1}e^{-\overline{\overline{J}}\theta} = \overline{\overline{G}}(\frac{\alpha}{\alpha^2+\gamma^2}, \frac{1}{\beta}, \frac{-\gamma}{\alpha^2+\gamma^2})$$

Eigenvalues of a gyrotropic dyadic

- Basic properties of the two-dimensional dyadics $\overline{\overline{I}}_t, \overline{\overline{J}}$:

$$\overline{\overline{I}}_t {}^\times_\times \mathbf{uu} = \overline{\overline{I}}_t, \quad \overline{\overline{J}} {}^\times_\times \mathbf{uu} = \overline{\overline{J}}, \quad \overline{\overline{I}}_t {}^\times_\times \overline{\overline{J}} = 0,$$

$$\overline{\overline{J}} : \overline{\overline{J}} = 2, \quad \overline{\overline{I}}_t : \overline{\overline{I}}_t = 2, \quad \overline{\overline{I}}_t : \overline{\overline{J}} = 0, \quad \overline{\overline{I}}_t^{(2)} = \overline{\overline{J}}^{(2)} = \mathbf{uu}$$

- lead to the following properties of the gyrotropic dyadic

$$\overline{\overline{G}}^{(2)}(\alpha, \beta, \gamma) = \overline{\overline{G}}(\alpha\beta, \alpha^2+\gamma^2, \beta\gamma), \quad \mathrm{spm}\overline{\overline{G}} = \mathrm{tr}\overline{\overline{G}}^{(2)} = 2\alpha\beta+\alpha^2+\gamma^2,$$

$$\det\overline{\overline{G}}(\alpha, \beta, \gamma) = \beta(\alpha^2+\gamma^2), \qquad \mathrm{tr}\overline{\overline{G}}(\alpha, \beta, \gamma) = 2\alpha+\beta$$

- Eigenvalues from the characteristic equation

$$\det(\overline{\overline{G}}-\lambda\overline{\overline{I}}) = \det\overline{\overline{G}} - \mathrm{spm}\overline{\overline{G}}\,\lambda + \mathrm{tr}\overline{\overline{G}}\,\lambda^2 - \lambda^3 = -(\lambda-\beta)((\lambda-\alpha)^2+\gamma^2) = 0$$

- solutions $\lambda_1 = \beta, \quad \lambda_{2,3} = \alpha \pm j\gamma$
- Special case uniaxial medium: $\gamma = 0 \quad \Rightarrow \quad \lambda_2 = \lambda_3 = \alpha$

Eigenvectors of a gyrotropic dyadic

- Eigenvector $\mathbf{a}_1$ corresponding to $\lambda_1 = \beta$ obtained from

$$(\overline{\overline{G}} - \lambda_1 \overline{\overline{I}})^{(2)} = ((\alpha-\beta)\overline{\overline{I}}_t + \gamma\overline{\overline{J}})^{(2)} = ((\alpha-\beta)^2+\gamma^2)\mathbf{uu}, \ \Rightarrow \ \mathbf{a}_1 = \mathbf{u},$$

- Eigenvectors $\mathbf{a}_{2,3}$ corresponding to $\lambda_{2,3} = \alpha \pm j\gamma$ with $\gamma \neq 0$ obtained from

$$(\overline{\overline{G}} - \lambda_{2,3}\overline{\overline{I}})^{(2)} = \mp j\gamma(\beta - \alpha \mp j\gamma)(\overline{\overline{I}}_t \pm j\overline{\overline{J}})$$

- Expanding dyadics $\overline{\overline{I}}_t \pm j\overline{\overline{J}}$ in an ONB $(\mathbf{v}, \mathbf{w}, \mathbf{u} = \mathbf{v} \times \mathbf{w})$ as

$$\overline{\overline{I}}_t \pm j(\mathbf{v} \times \mathbf{w}) \times \overline{\overline{I}} = \mathbf{vv} + \mathbf{ww} \pm j(\mathbf{wv} - \mathbf{vw}) = (\mathbf{v} \pm j\mathbf{w})(\mathbf{v} \mp j\mathbf{w})$$

$$\Rightarrow \quad (\overline{\overline{G}} - \lambda_{2,3}\overline{\overline{I}})^{(2)} = \mp j\gamma(\beta - \alpha \mp j\gamma)(\mathbf{v} \pm j\mathbf{w})(\mathbf{v} \mp j\mathbf{w})$$

- Eigenvectors $\mathbf{a}_{2,3} = \mathbf{v} \mp j\mathbf{w} = (\overline{\overline{I}} \mp j\overline{\overline{J}}) \cdot \mathbf{v}$ are circularly polarized
  $\mathbf{v}$ can be chosen as any vector $\perp \mathbf{u}$

Problems

2.1 Derive the inverse of the dyadic $\mathbf{ab} + \mathbf{c} \times \overline{\overline{I}}$ through vector algebra by solving the following linear equation for the vector $\mathbf{x}$:

$$\mathbf{a}(\mathbf{b} \cdot \mathbf{x}) + \mathbf{c} \times \mathbf{x} = \mathbf{y}, \quad \mathbf{x} = ?$$

2.2 Derive the two-dimensional inverse of the dyadic $\overline{\overline{A}} = \mathbf{ab} + \mathbf{cd}$ whose vectors $\mathbf{a} \cdots \mathbf{d}$ are orthogonal to a unit vector $\mathbf{n}$ by studying the expression $\overline{\overline{A}} \cdot (\overline{\overline{A}}^T {}^\times_\times \mathbf{nn})$ The dyadic $\overline{\overline{I}} - \mathbf{nn}$ serves as the two-dimensional unit dyadic.

2.3 Assuming that the inverse of the dyadic $\overline{\overline{A}}$ is known, find an expression for the inverse of the dyadic $\overline{\overline{B}} = \overline{\overline{A}} + \mathbf{ab}$ and check the result.

S-96.510 Advanced Field Theory
3. Basic Electromagnetic Equations

I.V.Lindell

Electromagnetic quantities

- Sinusoidal time dependence $e^{j\omega t}$ assumed
- Electric field intensity $\mathbf{E}$
- Magnetic field intensity $\mathbf{H}$
- Electric flux density $\mathbf{D}$
- Magnetic flux density $\mathbf{B}$
- Electric current density $\mathbf{J}$
- Magnetic current density $\mathbf{M}$
- Linear, time-invariant media, medium equations

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

Classification of electromagnetic media

- Medium dyadics for different types of media:
- Isotropic $\overline{\overline{\epsilon}} = \epsilon\overline{\overline{I}}, \quad \overline{\overline{\xi}} = 0, \quad \overline{\overline{\zeta}} = 0, \quad \overline{\overline{\mu}} = \mu\overline{\overline{I}}$
- Bi-isotropic $\overline{\overline{\epsilon}} = \epsilon\overline{\overline{I}}, \quad \overline{\overline{\xi}} = \xi\overline{\overline{I}}, \quad \overline{\overline{\zeta}} = \zeta\overline{\overline{I}}, \quad \overline{\overline{\mu}} = \mu\overline{\overline{I}}$
- Anisotropic $\overline{\overline{\epsilon}}, \quad \overline{\overline{\xi}} = 0, \quad \overline{\overline{\zeta}} = 0, \quad \overline{\overline{\mu}}$
- Bi-anisotropic $\overline{\overline{\epsilon}}, \quad \overline{\overline{\xi}}, \quad \overline{\overline{\zeta}}, \quad \overline{\overline{\mu}}$
- Homogeneous: constant medium dyadics
- Inhomogeneous: $\overline{\overline{\epsilon}}(\mathbf{r}), \cdots$
- Dispersive: $\overline{\overline{\epsilon}}(\omega), \cdots$
- Lossless, reciprocal, etc...

Examples of artificial media

- Medium dyadics can be realized through microscopic elements producing electric and magnetic dipole moments

$$\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}_e, \quad \mathbf{B} = \mu_o \mathbf{H} + \mathbf{P}_m$$

$$\mathbf{P}_e = \overline{\overline{\chi}}_{ee} \cdot \mathbf{E} + \overline{\overline{\chi}}_{em} \cdot \mathbf{H}, \quad \mathbf{P}_m = \overline{\overline{\chi}}_{me} \cdot \mathbf{E} + \overline{\overline{\chi}}_{mm} \cdot \mathbf{H}$$

- Metal needles $\overline{\overline{\chi}}_{em} = \overline{\overline{\chi}}_{me} = \overline{\overline{\chi}}_{mm} = 0$, dielectric medium
- Metal rings $\overline{\overline{\chi}}_{em} = \overline{\overline{\chi}}_{me} = 0$, dielectric-magnetic medium
- Metal helices, $\mathbf{P}_e$ and $\mathbf{P}_m$ parallel, chiral medium
- Omega-shaped particles, $\mathbf{P}_e \cdot \mathbf{P}_m = 0$, omega medium

The Maxwell Equations

- For time-harmonic fields only curl equations needed.
  Homogeneous medium assumed: constant parameter dyadics

- Isotropic

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\epsilon\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r})$$

$$-\nabla \times \mathbf{E}(\mathbf{r}) = j\omega\mu\mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})$$

- Anisotropic

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\overline{\overline{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r})$$

$$-\nabla \times \mathbf{E}(\mathbf{r}) = j\omega\overline{\overline{\mu}} \cdot \mathbf{H}(\mathbf{r}) + \mathbf{M}(\mathbf{r})$$

- Bi-anisotropic

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\overline{\overline{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) + j\omega\overline{\overline{\xi}} \cdot \mathbf{H}(\mathbf{r}) + \mathbf{J}(\mathbf{r})$$

$$-\nabla \times \mathbf{E}(\mathbf{r}) = j\omega\overline{\overline{\mu}} \cdot \mathbf{H}(\mathbf{r}) + j\omega\overline{\overline{\zeta}} \cdot \mathbf{E}(\mathbf{r}) + \mathbf{M}(\mathbf{r})$$

Operator notation

- Maxwell equations in terms of six-vectors and six-dyadics

$$\begin{pmatrix} 0 & \nabla\times\overline{\overline{I}} \\ -\nabla\times\overline{\overline{I}} & 0 \end{pmatrix}\cdot\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} - j\omega\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}\cdot\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ \mathbf{M} \end{pmatrix}$$

- More compact notation through the Maxwell six-dyadic operator

$$\mathsf{L}(\nabla)\cdot\mathsf{e}(\mathbf{r}) = \mathsf{j}(\mathbf{r}),$$

$$\mathsf{L}(\nabla) = \begin{pmatrix} -j\omega\overline{\overline{\epsilon}} & (\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \\ -(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) & -j\omega\overline{\overline{\mu}} \end{pmatrix}$$

- Electric and magnetic fields are coupled in the Maxwell equations decoupling through elimination or operator diagonalization

Diagonalization of operators

- Decoupling through adjoint Maxwell operators

$$\mathsf{L}^a(\nabla) = \begin{pmatrix} j\omega\overline{\overline{I}} & (\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}})\cdot\overline{\overline{\mu}}^{-1} \\ -(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}})\cdot\overline{\overline{\epsilon}}^{-1} & j\omega\overline{\overline{I}} \end{pmatrix}$$

$$\mathsf{L}_a(\nabla) = \begin{pmatrix} j\omega\overline{\overline{I}} & \overline{\overline{\epsilon}}^{-1}\cdot(\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \\ -\overline{\overline{\mu}}^{-1}\cdot(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) & j\omega\overline{\overline{I}} \end{pmatrix}$$

- Diagonalization of the six-dyadic operator

$$\mathsf{L}^a(\nabla)\cdot\mathsf{L}(\nabla) = \mathsf{L}(\nabla)\cdot\mathsf{L}_a(\nabla) = \begin{pmatrix} \overline{\overline{H}}_e(\nabla) & 0 \\ 0 & \overline{\overline{H}}_m(\nabla) \end{pmatrix}$$

- Dyadic Helmholtz operators $\overline{\overline{H}}_e(\nabla)$, $\overline{\overline{H}}_m(\nabla)$

Helmholtz operators

- Dyadic Helmholtz operators are of second order

$$\overline{\overline{H}}_e(\nabla) = -(\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \overline{\overline{\mu}}^{-1} \cdot (\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) + \omega^2\overline{\overline{\epsilon}}$$

$$\overline{\overline{H}}_m(\nabla) = -(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot (\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) + \omega^2\overline{\overline{\mu}}$$

- Special cases: anisotropic medium

$$\overline{\overline{H}}_e(\nabla) = \overline{\overline{\mu}}^{-1} {}^{\times}_{\times} \nabla\nabla + \omega^2\overline{\overline{\epsilon}}, \quad \overline{\overline{H}}_m(\nabla) = \overline{\overline{\epsilon}}^{-1} {}^{\times}_{\times} \nabla\nabla + \omega^2\overline{\overline{\mu}}$$

- Bi-isotropic medium, $k = \omega\sqrt{\mu\epsilon}$

$$\overline{\overline{H}}(\nabla) = \mu\overline{\overline{H}}_e(\nabla) = \epsilon\overline{\overline{H}}_m(\nabla) = -(\nabla\times\overline{\overline{I}} - j\omega\xi\overline{\overline{I}})\cdot(\nabla\times\overline{\overline{I}} + j\omega\zeta\overline{\overline{I}}) + k^2\overline{\overline{I}}$$

- Isotropic medium, $k^2 = \omega^2\mu\epsilon$

$$\overline{\overline{H}}(\nabla) = \mu\overline{\overline{H}}_e(\nabla) = \epsilon\overline{\overline{H}}_m(\nabla) = \overline{\overline{I}} {}^{\times}_{\times} \nabla\nabla + k^2\overline{\overline{I}}$$

Helmholtz equations

- Operating the Maxwell equations by the adjoint operator $L^a(\nabla)$

$$\mathsf{L}^a(\nabla)\cdot\mathsf{L}(\nabla)\cdot\mathsf{e}(\mathbf{r}) = \mathsf{L}^a(\nabla)\cdot\mathsf{j}(\mathbf{r})$$

- gives the Helmholtz equations with dyadic operators (2nd order)

$$\overline{\overline{H}}_e(\nabla)\cdot\mathbf{E}(\mathbf{r}) = j\omega\mathbf{J}(\mathbf{r}) + (\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}})\cdot\overline{\overline{\mu}}^{-1}\cdot\mathbf{M}(\mathbf{r}),$$

$$\overline{\overline{H}}_m(\nabla)\cdot\mathbf{H}(\mathbf{r}) = j\omega\mathbf{M}(\mathbf{r}) - (\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}})\cdot\overline{\overline{\epsilon}}^{-1}\cdot\mathbf{J}(\mathbf{r})$$

- Operating by $\overline{\overline{H}}_{e,m}^{(2)T}(\nabla)\cdot$ gives equations with scalar Helmholtz determinant operators $\det\overline{\overline{H}}_{e,m}(\nabla)$ (4th order)

$$\det\overline{\overline{H}}_e(\nabla)\ \mathbf{E}(\mathbf{r}) = \overline{\overline{H}}_e^{(2)T}(\nabla)\cdot[j\omega\mathbf{J}(\mathbf{r}) + (\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}})\cdot\overline{\overline{\mu}}^{-1}\cdot\mathbf{M}(\mathbf{r})]$$

$$\det\overline{\overline{H}}_m(\nabla)\ \mathbf{H}(\mathbf{r}) = \overline{\overline{H}}_m^{(2)T}(\nabla)\cdot[j\omega\mathbf{M}(\mathbf{r}) - (\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}})\cdot\overline{\overline{\epsilon}}^{-1}\cdot\mathbf{J}(\mathbf{r})]$$

Potentials

- Expressing the field six-dyadic as $\mathsf{e}(\mathbf{r}) = \mathsf{L}_a(\nabla)\cdot\mathsf{f}(\mathbf{r})$ in terms of two vector potentials $\mathsf{f} = (\mathbf{F}\ \mathbf{G})$

$$\mathbf{E}(\mathbf{r}) = j\omega\mathbf{F}(\mathbf{r}) + \overline{\overline{\epsilon}}^{-1}\cdot(\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}})\cdot\mathbf{G}(\mathbf{r}),$$

$$\mathbf{H}(\mathbf{r}) = -\overline{\overline{\mu}}^{-1}\cdot(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}})\cdot\mathbf{F}(\mathbf{r}) + j\omega\mathbf{G}(\mathbf{r}),$$

- leads to Helmholtz equations with decoupled sources:

$$\mathsf{L}(\nabla)\cdot\mathsf{L}_a(\nabla)\cdot\mathsf{f}(\mathbf{r}) = \mathsf{j}(\mathbf{r})$$

$$\overline{\overline{H}}_e(\nabla)\cdot\mathbf{F}(\mathbf{r}) = \mathbf{J}(\mathbf{r}), \quad \overline{\overline{H}}_m(\nabla)\cdot\mathbf{G}(\mathbf{r}) = \mathbf{M}(\mathbf{r})$$

- Helmholtz determinant equations (4th order)

$$\mathrm{det}\overline{\overline{H}}_e(\nabla)\,\mathbf{F}(\mathbf{r}) = j\omega\overline{\overline{H}}_e^{(2)T}(\nabla)\cdot\mathbf{J}(\mathbf{r}), \quad \mathrm{det}\overline{\overline{H}}_m(\nabla)\,\mathbf{G}(\mathbf{r}) = j\omega\overline{\overline{H}}_m^{(2)T}(\nabla)\cdot\mathbf{M}(\mathbf{r})$$

Helmholtz operators for isotropic medium 1

- Helmholtz dyadic operator

$$\overline{\overline{H}}(\nabla) = \overline{\overline{I}}{}^\times_\times \nabla\nabla + k^2\overline{\overline{I}} = (\nabla^2 + k^2)\overline{\overline{I}} - \nabla\nabla$$

- Helmholtz adjoint operator

$$\overline{\overline{H}}{}^{(2)T}(\nabla) = \frac{1}{2}(\overline{\overline{I}}{}^\times_\times \nabla\nabla + k^2\overline{\overline{I}}){}^\times_\times(\overline{\overline{I}}{}^\times_\times \nabla\nabla + k^2\overline{\overline{I}})$$

$$= \nabla^2\nabla\nabla + k^2(2\nabla^2\overline{\overline{I}} - \overline{\overline{I}}{}^\times_\times \nabla\nabla) + k^4\overline{\overline{I}} = (\nabla^2 + k^2)(\nabla\nabla + k^2\overline{\overline{I}})$$

- Helmholtz determinant operator

$$\mathrm{det}\overline{\overline{H}}(\nabla) = \frac{1}{3}\overline{\overline{H}}{}^{(2)}(\nabla) : \overline{\overline{H}}(\nabla) = k^2(\nabla^2 + k^2)^2$$

Helmholtz equation for isotropic medium 2

- Helmholtz determinant equation (fourth order)

$$\mu^3 \mathrm{det}\overline{\overline{H}}_e(\nabla)\mathbf{E}(\mathbf{r}) = \mathrm{det}\overline{\overline{H}}(\nabla)\mathbf{E}(\mathbf{r}) = k^2(\nabla^2 + k^2)^2\mathbf{E}(\mathbf{r})$$

$$= \mu^2 \overline{\overline{H}}_e^{(2)T}(\nabla) \cdot [j\omega\mu\mathbf{J}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})]$$

$$= (\nabla^2 + k^2)(\nabla\nabla + k^2\overline{\overline{I}}) \cdot [j\omega\mu\mathbf{J}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})]$$

- reduces to a second-order equation (uniqueness of solution assumed!)

$$(\nabla^2 + k^2)\mathbf{E}(\mathbf{r}) = (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla) \cdot [j\omega\mu\mathbf{J}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})]$$

- Simpler equation for vector potential $\mathbf{E} = -j\omega(\overline{\overline{I}} + \nabla\nabla/k^2) \cdot \mathbf{A}(\mathbf{r})$

$$(\nabla^2 + k^2)\mathbf{A}(\mathbf{r}) = -\mu\mathbf{J}(\mathbf{r}) + \frac{j}{\omega}\nabla \times \mathbf{M}(\mathbf{r})$$

Helmholtz operators for bi-isotropic medium 1

- Denote $\xi = (\chi_r - j\kappa_r)\sqrt{\mu\epsilon}$, $\zeta = (\chi_r + j\kappa_r)\sqrt{\mu\epsilon}$
  κ_r= relative chirality parameter, χ_r= relative Tellegen parameter
- Dyadic Helmholtz operators factorizable in bi-isotropic medium:

$$\mu\overline{\overline{H}}_e(\nabla) = \epsilon\overline{\overline{H}}_m(\nabla) = \overline{\overline{H}}(\nabla) = -(\nabla \times \overline{\overline{I}} - k\kappa_r\overline{\overline{I}})^2 + k^2(1-\chi_r^2)\overline{\overline{I}}$$

$$= -\overline{\overline{L}}_+(\nabla) \cdot \overline{\overline{L}}_-(\nabla) = -\overline{\overline{L}}_-(\nabla) \cdot \overline{\overline{L}}_+(\nabla)$$

- Two auxiliary first-order operators $\overline{\overline{L}}_\pm(\nabla)$

$$\overline{\overline{L}}_+(\nabla) = \nabla \times \overline{\overline{I}} - k_+\overline{\overline{I}}, \quad \overline{\overline{L}}_-(\nabla) = \nabla \times \overline{\overline{I}} + k_-\overline{\overline{I}}$$

$$k_+ = (\sqrt{1-\chi_r^2} + \kappa_r)k, \quad k_- = (\sqrt{1-\chi_r^2} - \kappa_r)k$$

- Special case: isotropic medium, $k_\pm = k$, $\overline{\overline{L}}_\pm(\nabla) = \nabla \times \overline{\overline{I}} \mp k\overline{\overline{I}}$

Helmholtz operators for bi-isotropic medium 2

- Auxiliary operators can be evaluated as

$$\overline{\overline{L}}{}_{\pm}^{(2)}(\nabla) = \nabla\nabla \mp k_{\pm}\nabla \times \overline{\overline{I}} + k_{\pm}^2\overline{\overline{I}}, \qquad \det\overline{\overline{L}}_{\pm}(\nabla) = \mp k_{\pm}(\nabla^2 + k_{\pm}^2)$$

- Product can be expanded as a sum

$$\overline{\overline{L}}{}_{+}^{(2)}(\nabla)\cdot\overline{\overline{L}}{}_{-}^{(2)}(\nabla) = k_{-}\frac{\nabla^2 + k_{-}^2}{k_{+} + k_{-}}\overline{\overline{L}}{}_{+}^{(2)}(\nabla) + k_{+}\frac{\nabla^2 + k_{+}^2}{k_{+} + k_{-}}\overline{\overline{L}}{}_{-}^{(2)}(\nabla)$$

- Using $(\overline{\overline{A}}\cdot\overline{\overline{B}})^{(2)} = \overline{\overline{A}}{}^{(2)}\cdot\overline{\overline{B}}{}^{(2)}$ and $\det(\overline{\overline{A}}\cdot\overline{\overline{B}}) = \det\overline{\overline{A}}\,\det\overline{\overline{B}}$ gives

$$[\mu\overline{\overline{H}}_e(\nabla)]^{(2)} = k_{-}\frac{\nabla^2 + k_{-}^2}{k_{+} + k_{-}}\overline{\overline{L}}{}_{+}^{(2)}(\nabla) + k_{+}\frac{\nabla^2 + k_{+}^2}{k_{+} + k_{-}}\overline{\overline{L}}{}_{-}^{(2)}(\nabla)$$

$$\det(\mu\overline{\overline{H}}_e(\nabla)) = k_{+}k_{-}(\nabla^2 + k_{+}^2)(\nabla^2 + k_{-}^2)$$

Helmholtz equation for bi-isotropic medium

- The Helmholtz equations have the form

$$\mu\overline{\overline{H}}_e(\nabla)\cdot\mathbf{E}(\mathbf{r}) = j\omega\mu\mathbf{J}(\mathbf{r}) + (\nabla\times\overline{\overline{I}} - j\omega\xi\overline{\overline{I}})\cdot\mathbf{M}(\mathbf{r}) = \mathbf{g}(\mathbf{r})$$

$$\det[\mu\overline{\overline{H}}_e(\nabla)]\mathbf{E}(\mathbf{r}) = [\mu\overline{\overline{H}}_e(\nabla)]^{(2)T}\cdot\mathbf{g}(\mathbf{r})$$

- The Helmholtz determinant equation can be expanded to

$$(\nabla^2+k_+^2)(\nabla^2+k_-^2)\mathbf{E}(\mathbf{r}) = [\frac{\nabla^2+k_-^2}{k_+(k_++k_-)}\overline{\overline{L}}_+^{(2)T}(\nabla)+\frac{\nabla^2+k_+^2}{k_-(k_++k_-)}\overline{\overline{L}}_-^{(2)T}(\nabla)]\cdot\mathbf{g}(\mathbf{r})$$

- Field can be solved in two parts as $\mathbf{E}=\mathbf{E}_+ + \mathbf{E}_-$ from

$$(\nabla^2+k_\pm^2)\mathbf{E}_\pm(\mathbf{r}) = \frac{1}{k_\pm(k_++k_-)}\overline{\overline{L}}_\pm^{(2)T}(\nabla)\cdot\mathbf{g}(\mathbf{r})$$

- Two second-order equations for the bi-isotropic medium! General fourth-order equation (bi-anisotropic medium) does not reduce.

Electromagnetic problems

- Basic problem: find field from given source in homogeneous medium
- Solution in integral form if field from point source is known
- Field from point source → dyadic Green function $\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')$

$$\mathbf{E}(\mathbf{r}) = \int_V \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')dV' \quad \mathbf{r} \notin V$$

- Problems involving inhomogeneous media and/or boundaries often handled through integral equations

$$\int_V \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')dV' = \mathbf{E}(\mathbf{r}) = \text{known}$$

- Unknown = (equivalent) source $\mathbf{J}(\mathbf{r})$ in certain region V
- Problems of uniqueness due to non-radiating sources

Discontinuities in fields 1

- Heaviside unit step $U(z)$ generates the delta function $\delta(z)$

$$U(z) = 0,\ z < 0, \quad U(z) = 1,\ z > 0, \quad \Rightarrow \quad \partial_z U(z) = \delta(z)$$

- Step discontinuity in function $F(\mathbf{r})$ at surface S separating regions V_1 and V_2 generates the surface delta function $\delta_s(\mathbf{r})$

$$F(\mathbf{r}) = F_1,\ \mathbf{r} \in V_1, \quad F(\mathbf{r}) = F_2,\ \mathbf{r} \in V_2,$$

$$\nabla F(\mathbf{r}) = \mathbf{n}_2(F_2 - F_1)\delta_s(\mathbf{r}) = \mathbf{n}_1(F_1 - F_2)\delta_s(\mathbf{r}) = (\mathbf{n}_2 F_2 + \mathbf{n}_1 F_1)\delta_s(\mathbf{r})$$

- Integration of surface delta $\int_V g(\mathbf{r})\delta_s(\mathbf{r})dV = \int_S g(\mathbf{r})dS$

- If $f(\mathbf{r})$ discontinuous on surface S:

$$\nabla f(\mathbf{r}) = \{\nabla f(\mathbf{r})\}_{\rm cont} + (\nabla_s f(\mathbf{r}))\delta_s(\mathbf{r}), \quad \nabla_s f(\mathbf{r}) = \mathbf{n}_1 f(\mathbf{r}_1) + \mathbf{n}_2 f(\mathbf{r}_2)$$

- cont = no delta discontinuity, ∇_s = surface nabla operator

Discontinuities in fields 2

- Surface gradient, divergence and curl:

$$\nabla_s f = \mathbf{n}_1 f(\mathbf{r}_1) + \mathbf{n}_2 f(\mathbf{r}_2)$$
$$\nabla_s \cdot \mathbf{f} = \mathbf{n}_1 \cdot \mathbf{f}(\mathbf{r}_1) + \mathbf{n}_2 \cdot \mathbf{f}(\mathbf{r}_2)$$
$$\nabla_s \times \mathbf{f} = \mathbf{n}_1 \times \mathbf{f}(\mathbf{r}_1) + \mathbf{n}_2 \times \mathbf{f}(\mathbf{r}_2)$$

- δ_s-discontinuous sources create step-discontinuous fields:

$$\nabla \times \mathbf{E} = \{\nabla \times \mathbf{E}\}_{\rm cont} + (\nabla_s \times \mathbf{E})\delta_s(\mathbf{r}) = -j\omega\mathbf{B} - \{\mathbf{M}\}_{\rm cont} - \mathbf{M}_s\delta_s(\mathbf{r})$$
$$\nabla \times \mathbf{H} = \{\nabla \times \mathbf{H}\}_{\rm cont} + (\nabla_s \times \mathbf{H})\delta_s(\mathbf{r}) = j\omega\mathbf{D} + \{\mathbf{J}\}_{\rm cont} + \mathbf{J}_s\delta_s(\mathbf{r})$$

- Equating δ_s-discontinuous terms gives conditions on interfaces

$$\nabla_s \times \mathbf{E} = \mathbf{n}_1 \times \mathbf{E}_1 + \mathbf{n}_2 \times \mathbf{E}_2 = -\mathbf{M}_s$$
$$\nabla_s \times \mathbf{H} = \mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2 = \mathbf{J}_s$$
$$\nabla_s \cdot \mathbf{D} = \mathbf{n}_1 \cdot \mathbf{D}_1 + \mathbf{n}_2 \cdot \mathbf{D}_2 = \varrho_s$$
$$\nabla_s \cdot \mathbf{B} = \mathbf{n}_1 \cdot \mathbf{B}_1 + \mathbf{n}_2 \cdot \mathbf{B}_2 = \varrho_{ms}$$

Boundary conditions 1

- Boundary: fields vanish on side 2 of surface S

$$\nabla_s \times \mathbf{E} = \mathbf{n}_1 \times \mathbf{E}_1 = -\mathbf{M}_s, \quad \nabla_s \times \mathbf{H} = \mathbf{n}_1 \times \mathbf{H}_1 = \mathbf{J}_s$$

$$\nabla_s \cdot \mathbf{D} = \mathbf{n}_1 \cdot \mathbf{D}_1 = \varrho_s, \quad \nabla_s \cdot \mathbf{B} = \mathbf{n}_1 \cdot \mathbf{B}_1 = \varrho_{ms}$$

- Boundary relation between $\mathbf{J}_s, \mathbf{M}_s$. E.g., linear relation:

$$\mathbf{M}_s = -\mathbf{n}_1 \times \overline{\overline{Z}}_s \cdot \mathbf{J}_s, \quad \text{or} \quad \mathbf{E}_{1t} = \overline{\overline{Z}}_s \cdot \mathbf{J}_s$$

- Impedance boundary condition between tangential field components, $\overline{\overline{Z}}_s$= two-dimensional dyadic

$$\mathbf{E}_t = \overline{\overline{Z}}_s \cdot (\mathbf{n} \times \mathbf{H}_t), \quad \Rightarrow \quad \mathbf{H}_t = -(\mathbf{nn}{}^{\times}_{\times}\overline{\overline{Z}}_s)^{-1} \cdot (\mathbf{n} \times \mathbf{E}_t) = -(\mathrm{spm}\overline{\overline{Z}}_s)^{-1}\overline{\overline{Z}}{}_s^T \cdot (\mathbf{n} \times \mathbf{E}_t)$$

- Two-dimensional inverse! (spm = two-dimensional determinant)

$$\overline{\overline{A}}_t \cdot (\overline{\overline{A}}{}_t^T {}^{\times}_{\times} \mathbf{nn}) = (\mathrm{spm}\overline{\overline{A}}_t)\overline{\overline{I}}_t, \quad \overline{\overline{A}}{}_t^{-1} = \overline{\overline{A}}{}_t^T {}^{\times}_{\times} \mathbf{nn}/\mathrm{spm}\overline{\overline{A}}_t$$

Boundary conditions 2

- Local tangential ONB $\{\mathbf{v}, \mathbf{w} = \mathbf{n} \times \mathbf{v}\}$ on S

$$\overline{\overline{Z}}_s = Z_{vv}\mathbf{vv} + Z_{vw}\mathbf{vw} + Z_{wv}\mathbf{wv} + Z_{ww}\mathbf{ww}$$

- Isotropic impedance surface, perfect electric/magnetic conductor:

$$\overline{\overline{Z}}_s = Z_s\overline{\overline{I}}_t = Z_s(\overline{\overline{I}} - \mathbf{nn}), \quad Z_s = 0, \text{ (PEC)}, \quad Z_s = \infty, \text{ (PMC)}$$

- Self-dual impedance surface

$$\overline{\overline{Z}}_s = Z(\alpha\mathbf{vv} + \alpha^{-1}\mathbf{vv}{}^{\times}_{\times}\mathbf{nn}) = Z(\alpha\mathbf{vv} + \alpha^{-1}\mathbf{ww})$$

- Limit $\alpha \to 0$ = soft-and-hard surface (SHS) = perfect anisotropic surface, gives symmetric boundary conditions $\mathbf{v} \cdot \mathbf{E} = 0$, $\mathbf{v} \cdot \mathbf{H} = 0$

- Generalized SHS boundary (conditions $\mathbf{a} \cdot \mathbf{E} = 0$, $\mathbf{b} \cdot \mathbf{H} = 0$)

$$\overline{\overline{Z}}_s = Z(\alpha\mathbf{ba} + \alpha^{-1}\mathbf{ab}{}^{\times}_{\times}\mathbf{nn}), \quad \mathbf{a} \cdot \mathbf{b} = 1, \quad \alpha \to 0$$

Interface conditions

- Interface S between regions 1,2. Conditions for tangential fields

$$\begin{pmatrix} \mathbf{E}_{1t} \\ \mathbf{E}_{2t} \end{pmatrix} = \begin{pmatrix} \overline{\overline{Z}}_{11} & \overline{\overline{Z}}_{12} \\ \overline{\overline{Z}}_{21} & \overline{\overline{Z}}_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{n}_1 \times \mathbf{H}_1 \\ \mathbf{n}_2 \times \mathbf{H}_2 \end{pmatrix}$$

- For $\overline{\overline{Z}}_{12} = \overline{\overline{Z}}_{21}$ interface can be represented by a T circuit.
- Example: isotropic impedance sheet = thin layer of material on S

$$\mathbf{E}_{1t} = \mathbf{E}_{2t} = \mathbf{E}_t, \qquad \mathbf{J}_s = \mathbf{E}_t/Z_s = \mathbf{n}_1 \times \mathbf{H}_1 + \mathbf{n}_2 \times \mathbf{H}_2$$

- Circuit parameters $\overline{\overline{Z}}_{ij} = Z_s \overline{\overline{I}}_t \ \Rightarrow$ shunt element in the T-circuit.
- Example: thin dielectric layer, thickness $t \to 0$, $\epsilon \to \infty$
  assume finite impedance $0 < |(\epsilon_r - 1)k_o t| < \infty$

$$\mathbf{J}_s = j\omega(\epsilon - \epsilon_o)t\mathbf{E}_t, \qquad Z_s = \frac{1}{j\omega(\epsilon - \epsilon_o)t} = -j\frac{\eta_o}{(\epsilon_r - 1)k_o t}$$

Problems

3.1 Assuming uniaxial anisotropic medium with medium dyadics

$$\overline{\overline{\epsilon}} = \epsilon_t \overline{\overline{I}}_t + \epsilon_z \mathbf{u}_z\mathbf{u}_z \quad \overline{\overline{\mu}} = \mu_t \overline{\overline{I}}_t + \mu_z \mathbf{u}_z\mathbf{u}_z, \quad \overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z,$$

show that the Helmholtz determinant operator can be factorized as $\det\overline{\overline{H}}_e(\nabla) = H_1(\nabla)H_2(\nabla)$ where $H_i(\nabla)$ are two second-order scalar operators. Hint: you may need the expansion (prove it)

$$\overline{\overline{\epsilon}}^{(2)} {}^\times_\times \overline{\overline{\mu}}^{(2)} = \mu_t\epsilon_t(\epsilon_z\overline{\overline{\mu}} + \mu_z\overline{\overline{\epsilon}})$$

3.2 Assuming an anisotropic medium whose medium dyadics satisfy the relation $\overline{\overline{\mu}} = \tau\overline{\overline{\epsilon}}^T$ where τ is a scalar, show that the inverse of the Helmholtz dyadic operator can be expressed in the simple form

$$\overline{\overline{H}}_e^{-1}(\nabla) = \overline{\overline{L}}(\nabla)/L(\nabla),$$

where the dyadic operator $\overline{\overline{L}}(\nabla)$ and the scalar operator $L(\nabla)$ are both of the second order. Hint: use result of Problem 2.3.

S-96.510 Advanced Field Theory
4. Conditions for fields and media

I.V.Lindell

Uniqueness

- Linear differential equation and boundary conditions

$$L(\nabla)f(\mathbf{r}) = g(\mathbf{r}), \qquad B(\nabla)f(\mathbf{r}) = s(\mathbf{r}),$$

- If two solutions $f_1(\mathbf{r}), f_2(\mathbf{r})$

$$L(\nabla)[f_1(\mathbf{r}) - f_2(\mathbf{r})] = 0, \qquad B(\nabla)[f_1(\mathbf{r}) - f_2(\mathbf{r})] = 0$$

- Unique $f(\mathbf{r})$ if homogeneous (sourceless) problem has only the null solution

$$L(\nabla)f_o(\mathbf{r}) = 0, \quad B(\nabla)f_o(\mathbf{r}) = 0, \quad \Rightarrow \quad f_o(\mathbf{r}) = 0$$

- Homogeneous problem similar to an eigenvalue problem
- Uniqueness in integral equations more complicated because they involve sources as unknowns

Eigenvalue problem

- Example: sourceless Maxwell equations

$$\begin{pmatrix} 0 & \nabla\times\overline{\overline{I}} \\ -\nabla\times\overline{\overline{I}} & 0 \end{pmatrix}\cdot\begin{pmatrix} \mathbf{E}_o(\mathbf{r}) \\ \mathbf{H}_o(\mathbf{r}) \end{pmatrix} = j\omega\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}\cdot\begin{pmatrix} \mathbf{E}_o(\mathbf{r}) \\ \mathbf{H}_o(\mathbf{r}) \end{pmatrix}$$

- Of the form of an eigenvalue equation, ω=eigenvalue parameter

$$\mathsf{L}(\nabla)\cdot\mathsf{f}_o(\mathbf{r}) = j\omega\mathsf{M}\cdot\mathsf{f}_o(\mathbf{r}),$$

- Impedance conditions on surface S:

$$\mathbf{E}_{ot}(\mathbf{r}) = \overline{\overline{Z}}_s\cdot(\mathbf{n}\times\mathbf{H}_o(\mathbf{r})), \qquad \mathbf{r}\in S$$

- If $\omega = \omega_i$ in the eigenvalue spectrum $\{\omega_j\}$, solution $\mathsf{f}_o(\mathbf{r}) \neq 0$ multiple of the eigenvector $\mathsf{f}_{oi}(\mathbf{r})$, otherwise $\mathsf{f}_o(\mathbf{r}) = 0$

Example: Resonance cavity

- Closed PEC surface S with $\overline{\overline{Z}}_s = 0$
- Real resonance frequencies $\omega_1, \omega_2, \cdots$, resonance modes $\mathbf{E}_1(\mathbf{r}), \mathbf{E}_2(\mathbf{r}), \cdots$ satisfy the homogeneous equation + boundary conditions
- Mathematics: source problem nonunique at resonance $\omega = \omega_i$
- Physics: without losses fields become infinite at resonance
- Reality: losses make $\{\omega_i\}$ complex $\Rightarrow \omega \neq \omega_i$, uniqueness at real frequencies
- Practice: if losses small, trouble in numerical computation (almost nonunique due to roundoff errors)
- For some problems uniqueness can be proved through uniqueness theorems of the form $|\alpha| = \beta$, (β not positive real) $\Rightarrow \alpha = \beta = 0$

Uniqueness in electrostatics 1

- $f = \phi(\mathbf{r})$ potential, $g = -\varrho(\mathbf{r})/\epsilon$ charge, assume isotropic volume V bounded by surface S

$$\nabla^2 f = g, \quad \nabla^2 f_o = 0, \quad \text{in } V$$

$$0 = \int_V f_o^*(\nabla^2 f_o) dV = \int_V \nabla \cdot (f_o^* \nabla f_o) dV - \int_V |\nabla f_o|^2 dV$$

$$\text{Gauss' law} \Rightarrow \quad \int_V |\nabla f_o|^2 dV = \oint_S f_o^*(\mathbf{n} \cdot \nabla f_o) dS$$

- Different boundary conditions on S making the surface integral vanish will ensure uniqueness for the electrostatic field problem

$$\int_V |\nabla f_o|^2 dV = 0, \quad \Rightarrow \quad \nabla f_o = 0 \quad \Rightarrow \quad \text{field vanishes}$$

Uniqueness in electrostatics 2

- Boundary conditions for uniqueness

$$f_o = 0 \;\text{ on } S, \qquad f = s, \quad \text{Dirichlet condition}$$

$$\mathbf{n}\cdot\nabla f_o = 0 \;\text{ on } S, \qquad \mathbf{n}\cdot\nabla f = s, \quad \text{Neumann condition}$$

- Mixed Dirichlet (on S_1) and Neumann (on $S-S_1$) gives uniqueness
- What about condition $\alpha f_o + \beta\mathbf{n}\cdot\nabla f_o = 0$ on S ?

$$\oint_S |\alpha f_o+\beta\mathbf{n}\cdot\nabla f_o|^2 dS = |\alpha|^2\oint_S |f_o|^2 dS + |\beta|^2 \oint_S |\mathbf{n}\cdot\nabla f_o|^2 dS + 2\Re\{\alpha^*\beta\}\int_V |\nabla f_o|^2 dV = 0$$

- If $\Re\{\alpha^*\beta\} > 0$, $\Rightarrow \nabla f_o = 0$ in V
- $\Rightarrow$ Impedance boundary condition of the form $\alpha f + \beta\mathbf{n}\cdot\nabla f = s$ on S gives uniqueness for $\Re\{\alpha^*\beta\} > 0$

Uniqueness in electrodynamics (Example)

- Uniqueness for lossy isotropic medium with lossless boundary at real frequencies when $\mu\epsilon$ complex (complex resonance frequencies)

$$\oint_S \mathbf{n}\cdot\mathbf{E}_o\times\mathbf{H}_o^* dS = \int_V \nabla\cdot(\mathbf{E}_o\times\mathbf{H}_o^*)dV = -j\omega\mu\int_V |\mathbf{H}_o|^2 dV + j\omega\epsilon^*\int_V |\mathbf{E}_o|^2 dV$$

- Assume S ideal boundary $\Rightarrow \oint_S \mathbf{n}\cdot\mathbf{E}_o\times\mathbf{H}_o^* dS = 0$

$$\Im\{\mu\epsilon\} \neq 0, \quad \mu\epsilon\int_V |\mathbf{H}_o|^2 dV = |\epsilon|^2\int_V |\mathbf{E}_o|^2 dV \quad \Rightarrow \quad \mathbf{E}_o = 0, \mathbf{H}_o = 0$$

- Uniqueness for the resonator with lossless medium and boundaries only when frequency not one of eigenfrequencies, $\omega \neq \omega_i$.

Power conditions for medium parameters

- Complex Poynting vector for time-harmonic fields:

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$$

- $\Re\{\mathbf{S}\}$ = average power flow in the field [Watts/m^2]
- $P = \nabla \cdot \Re\{\mathbf{S}\}$ = average power created by medium [Watts/m^3]
- Active medium $P > 0$: Medium gives energy to field
- Passive medium $P < 0$: Field gives energy to medium
- Lossless medium $P = 0$: No exchange of energy
- Nature of power exchange depends on medium parameters

Power exchange

- Power created in the medium

$$P = \nabla\cdot\Re\{\mathbf{S}\} = \frac{1}{2}\Re\{(\nabla\times\mathbf{E})\cdot\mathbf{H}^* - (\nabla\times\mathbf{H})^*\cdot\mathbf{E}\}$$

$$= \frac{1}{2}\Re\{-j\omega\mathbf{B}\cdot\mathbf{H}^* + j\omega\mathbf{D}^*\cdot\mathbf{E}\} = \frac{\omega}{2}\Im\{\mathbf{E}^*\cdot\mathbf{D} + \mathbf{H}^*\cdot\mathbf{B}\}$$

- In six-vector notation

$$P = \frac{\omega}{2}\Im\{\mathsf{e}^*\cdot\mathsf{M}\cdot\mathsf{e}\}, \qquad \mathsf{e} = \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, \quad \mathsf{M} = \begin{pmatrix}\overline{\overline{\epsilon}} & \overline{\overline{\xi}}\\ \overline{\overline{\zeta}} & \overline{\overline{\mu}}\end{pmatrix}$$

- Lossless medium: $P = 0$ for all e
- Lossy (passive) medium: $P < 0$ for all e
- Active medium if $P > 0$ for some field e_1

Lossless bi-anisotropic medium

- Condition for lossless medium: $P = 0$ for all possible fields $\mathbf{E}, \mathbf{H}$

  $2j\Im\{\mathsf{e}^*\cdot\mathsf{M}\cdot\mathsf{e}\} = \mathsf{e}^*\cdot\mathsf{M}\cdot\mathsf{e} - \mathsf{e}\cdot\mathsf{M}^*\cdot\mathsf{e}^* = \mathsf{e}\cdot(\mathsf{M}^T - \mathsf{M}^*)\cdot\mathsf{e}^* = 0$ for all e

- To prove: $\mathsf{e}\cdot\mathsf{A}\cdot\mathsf{e}^* = 0$ for all e implies $\mathsf{A} = 0$
- Take $\mathsf{e} = \mathsf{a} + \mathsf{b} \;\Rightarrow\; \mathsf{a}\cdot\mathsf{A}\cdot\mathsf{b}^* + \mathsf{b}\cdot\mathsf{A}\cdot\mathsf{a}^* = 0$
- Take $\mathsf{e} = \mathsf{a} + j\mathsf{b} \;\Rightarrow\; \text{-j}\mathsf{a}\cdot\mathsf{A}\cdot\mathsf{b}^* + j\mathsf{b}\cdot\mathsf{A}\cdot\mathsf{a}^* = 0$
- Follows $\mathsf{a}\cdot\mathsf{A}\cdot\mathsf{b}^* = 0$ for all $\mathsf{a}, \mathsf{b} \Rightarrow \mathsf{A} = 0$
- Condition for lossless medium: M is a Hermitian six-dyadic

$$\mathsf{M}^T - \mathsf{M}^* = 0 \quad \Rightarrow \quad \mathsf{M}^{*T} = \mathsf{M}$$

Parameters of lossless medium

- Medium six-dyadic Hermitian for lossless media

$$\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}^{*T} = \begin{pmatrix} \overline{\overline{\epsilon}}^{*T} & \overline{\overline{\zeta}}^{*T} \\ \overline{\overline{\xi}}^{*T} & \overline{\overline{\mu}}^{*T} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}$$

$$\overline{\overline{\epsilon}}^{*T} = \overline{\overline{\epsilon}}, \quad \overline{\overline{\mu}}^{*T} = \overline{\overline{\mu}}, \quad \overline{\overline{\xi}}^{*T} = \overline{\overline{\zeta}},$$

- $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ Hermitian dyadics and $\overline{\overline{\xi}}, \overline{\overline{\zeta}}$ a Hermitian pair of dyadics
- Define $\overline{\overline{\xi}} = (\overline{\overline{\chi}} - j\overline{\overline{\kappa}})\sqrt{\mu_o\epsilon_o}, \quad \overline{\overline{\zeta}} = (\overline{\overline{\chi}} + j\overline{\overline{\kappa}})\sqrt{\mu_o\epsilon_o},$

$$\Rightarrow \quad \overline{\overline{\chi}}^{*T} = \overline{\overline{\chi}}, \quad \overline{\overline{\kappa}}^{*T} = \overline{\overline{\kappa}}$$

- Lossless bi-anisotropic medium: $\overline{\overline{\epsilon}}, \overline{\overline{\mu}}, \overline{\overline{\chi}}, \overline{\overline{\kappa}}$ are Hermitian dyadics

Examples of lossless media

- Bi-isotropic medium: $\epsilon, \mu, \chi, \kappa$ are all real for a lossless medium

$$\epsilon^* = \epsilon, \quad \mu^* = \mu, \quad \chi^* = \chi, \quad \kappa^* = \kappa$$

- Gyrotropic medium: $\overline{\overline{\epsilon}}, \overline{\overline{\mu}}, \overline{\overline{\chi}}, \overline{\overline{\kappa}}$ are of the general gyrotropic form $\overline{\overline{G}} = G_z \mathbf{u}_z \mathbf{u}_z + G_t \overline{\overline{I}}_t + G_g \mathbf{u}_z \times \overline{\overline{I}}$

- Lossless medium: gyrotropic dyadics are Hermitian $\overline{\overline{G}}^{*T} = \overline{\overline{G}}$

$$G_z^* = G_z, \quad G_t^* = G_t, \quad G_g^* = -G_g$$

- Hermitian gyrotropic dyadic in a form where G_z, G_t, G_g are real

$$\overline{\overline{G}} = G_z \mathbf{u}_z \mathbf{u}_z + G_t \overline{\overline{I}}_t + jG_g \mathbf{u}_z \times \overline{\overline{I}}$$

- General lossless medium: parameter dyadics $\overline{\overline{D}} = \overline{\overline{S}} + \mathbf{a} \times \overline{\overline{I}}$ with symmetric part $\overline{\overline{S}}$ real and antisymmetric part $\mathbf{a} \times \overline{\overline{I}}$ imaginary

Lossy medium

- Condition for lossy media: $\Re\{\nabla\cdot\mathbf{S}\}<0$ for any fields

  $$\Rightarrow \quad 2\Im\{\mathsf{e}^*\cdot\mathsf{M}\cdot\mathsf{e}\} = \mathsf{e}\cdot[(-j\mathsf{M})^T + (-j\mathsf{M})^*]\cdot\mathsf{e}^* < 0 \text{ for all } \mathsf{e}$$

- Hermitian dyadic H negative definite if $\mathsf{e}\cdot\mathsf{H}\cdot\mathsf{e}^*<0$ for all e
- $\Rightarrow \quad \mathsf{H} = (-j\mathsf{M})^T + (-j\mathsf{M})^*$ must be negative definite
- Take $\mathbf{H}=0$, $\Rightarrow\ j(\overline{\overline{\epsilon}}^* - \overline{\overline{\epsilon}}^T)$ must be negative definite
- Take $\mathbf{E}=0$, $\Rightarrow\ j(\overline{\overline{\mu}}^* - \overline{\overline{\mu}}^T)$ must be negative definite
- Relation for $\overline{\overline{\xi}}, \overline{\overline{\zeta}}$ complicated [see MOTL 29(3)175-178 May 2001]
- Isotropic medium: $\epsilon_{im}<0,\ \mu_{im}<0$
- Isotropic chiral medium: $|\kappa_{im}|^2 < |\epsilon_{im}|\ |\mu_{im}|/\mu_o\epsilon_o$ (obtained after some algebra). Note: if $\epsilon_{im}=0$ or $\mu_{im}=0 \Rightarrow \kappa_{im}=0$

Surface impedance conditions

- Impedance condition $\mathbf{E}_t = \overline{\overline{Z}}_s \cdot (\mathbf{n} \times \mathbf{H})$, or $\mathbf{n} \times \mathbf{H} = \overline{\overline{Y}}_s \cdot \mathbf{E}_t$
- Power flow from impedance surface to the field (direction $\mathbf{n}$)

$$P = \frac{1}{2}\Re\{\mathbf{n} \cdot \mathbf{E} \times \mathbf{H}^*\} = -\frac{1}{2}\Re\{\mathbf{E}^* \cdot \mathbf{n} \times \mathbf{H}\}$$

$$= -\frac{1}{2}\Re\{\mathbf{E}^* \cdot \overline{\overline{Y}}_s \cdot \mathbf{E}\} = -\frac{1}{2}\mathbf{E}^* \cdot \overline{\overline{Y}}_{sH} \cdot \mathbf{E}$$

- $\overline{\overline{Y}}_{sH} = \frac{1}{2}(\overline{\overline{Y}}_s + \overline{\overline{Y}}_s^{*T})$, Hermitian part of $\overline{\overline{Y}}_s$
- Lossless surface: $P = 0 \quad \Rightarrow \quad \overline{\overline{Y}}_{sH} = 0$, $\overline{\overline{Y}}_s$ antihermitian dyadic
- Example: $\overline{\overline{Y}}_s$ symmetric, $\overline{\overline{Y}}_s = j\overline{\overline{B}}_s$ imaginary (reactive surface)
- Lossy medium: $P < 0 \quad \Rightarrow$ Hermitian part of $\overline{\overline{Y}}_s$ positive definite
- E.g.: $\overline{\overline{Y}}_s$ symmetric, $\Rightarrow \overline{\overline{Y}}_s = \overline{\overline{G}}_s + j\overline{\overline{B}}_s$ with $\overline{\overline{G}}_s$ pos. definite

Ideal boundary

- Ideal boundary: Poynting vector has no normal component

$$\mathbf{n} \cdot \mathbf{S} = \frac{1}{2}\mathbf{n} \cdot \mathbf{E} \times \mathbf{H}^* = 0$$

- Ideal boundary is lossless. Examples: PEC, PMC and SHS
- Relation between tangential fields: $\mathbf{E}_t$ and $\mathbf{H}_t^*$ parallel
- **Anisotropic** ideal surface: exists (complex) tangential vector $\mathbf{a}$ such that

$$\mathbf{a} \cdot \mathbf{E} = 0, \quad \mathbf{a}^* \cdot \mathbf{H} = 0$$

- Generalized SHS, for $\mathbf{a} = \mathbf{a}^* = \mathbf{v}$ gives real classical SHS
- **Isotropic** ideal surface: exists a (complex) scalar Z such that

$$\mathbf{E}_t = Z\mathbf{H}_t^*, \quad \text{not a linear condition!}$$

Energy condition

- Lossless, nondispersive medium assumed (otherwise complicated)
- Condition: energy stored in the medium W positive for all fields

$$W = \frac{1}{4}(\mathbf{E}\ \mathbf{H}) \cdot \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}^* = \frac{1}{4}\mathsf{e} \cdot \mathsf{M} \cdot \mathsf{e}^* > 0$$

- M Hermitian, condition requires: M pos.definite $\Rightarrow \mathsf{M}^{-1}$ PD

$$\mathsf{M}^{-1} = \begin{pmatrix} (\overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}})^{-1} & -\overline{\overline{\epsilon}}^{-1} \cdot \overline{\overline{\xi}} \cdot (\overline{\overline{\mu}} - \overline{\overline{\zeta}} \cdot \overline{\overline{\epsilon}}^{-1} \cdot \overline{\overline{\xi}})^{-1} \\ -\overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}} \cdot (\overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}})^{-1} & (\overline{\overline{\mu}} - \overline{\overline{\zeta}} \cdot \overline{\overline{\epsilon}}^{-1} \cdot \overline{\overline{\xi}})^{-1} \end{pmatrix}$$

- $\Rightarrow$ dyadics $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$, $\overline{\overline{\epsilon}} - \overline{\overline{\xi}} \cdot \overline{\overline{\mu}}^{-1} \cdot \overline{\overline{\zeta}}$, $\overline{\overline{\mu}} - \overline{\overline{\zeta}} \cdot \overline{\overline{\epsilon}}^{-1} \cdot \overline{\overline{\xi}}$ must all be PD
- Example: bi-isotropic medium (lossless $\Rightarrow \epsilon, \mu, \chi, \kappa$ real)
  $\epsilon > 0, \quad \mu > 0, \qquad \xi\zeta < \mu\epsilon \quad \Rightarrow \quad \chi^2 + \kappa^2 < \mu\epsilon/\mu_o\epsilon_o$
- Condition limits magnitudes of χ, κ parameters

Reciprocity conditions 1

- Reaction of source and field defined as [Rumsey 1954]

$$< 1,2 >= \int (\mathbf{E}_1 \ \mathbf{H}_1) \cdot \begin{pmatrix} \mathbf{J}_2 \\ -\mathbf{M}_2 \end{pmatrix} dV$$

- Medium reciprocal when reaction is symmetric $< 1,2 >=< 2,1 >$

$$0 = \int (\mathbf{E}_1 \ \mathbf{H}_1) \cdot \begin{pmatrix} \mathbf{J}_2 \\ -\mathbf{M}_2 \end{pmatrix} dV - \int (\mathbf{E}_2 \ \mathbf{H}_2) \cdot \begin{pmatrix} \mathbf{J}_1 \\ -\mathbf{M}_1 \end{pmatrix} dV$$

- Replace sources from the Maxwell equations, apply Gauss' law

$$\oint_S \mathbf{n} \cdot (\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1) dS = j\omega \int_V (\mathbf{E}_1 \ \mathbf{H}_1) \cdot \begin{pmatrix} \overline{\overline{\epsilon}} - \overline{\overline{\epsilon}}^T & \overline{\overline{\xi}} + \overline{\overline{\zeta}}^T \\ -\overline{\overline{\zeta}} - \overline{\overline{\xi}}^T & -\overline{\overline{\mu}} + \overline{\overline{\mu}}^T \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_2 \\ \mathbf{H}_2 \end{pmatrix} dV$$

- Must be valid for any fields in any volume V bounded by S. Integrals must vanish separately ⇒ conditions for medium and boundary parameters.

Reciprocity conditions 2

- Volume integral vanishes for any fields under reciprocity conditions for the medium:

$$\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^T, \quad \overline{\overline{\xi}} = -\overline{\overline{\zeta}}^T, \quad \overline{\overline{\zeta}} = -\overline{\overline{\xi}}^T, \quad \overline{\overline{\mu}} = \overline{\overline{\mu}}^T$$

- For example, bi-isotropic medium reciprocal if $\xi = -\zeta, \;\Rightarrow\; \chi = 0$. Tellegen parameter χ = nonreciprocity parameter
- When is medium with gyrotropic dyadics reciprocal? $\overline{\overline{\epsilon}}, \overline{\overline{\mu}}$ symmetric $\Rightarrow \epsilon_g = \mu_g = 0$, $\overline{\overline{\xi}} = -\overline{\overline{\zeta}}^T \Rightarrow \xi_u = -\zeta_u$, $\xi_t = -\zeta_t$, $\xi_g = \zeta_g$
- Surface integral vanishes under reciprocity condition for symmetric surface impedance dyadic

$$\overline{\overline{Z}}_s = \overline{\overline{Z}}_s^T$$

Reciprocal and lossless media

- Conditions for a medium being both lossless and reciprocal:

$$\begin{pmatrix} \overline{\overline{\epsilon}}^{T*} & \overline{\overline{\zeta}}^{T*} \\ \overline{\overline{\xi}}^{T*} & \overline{\overline{\mu}}^{T*} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}}^{T} & -\overline{\overline{\zeta}}^{T} \\ -\overline{\overline{\xi}}^{T} & \overline{\overline{\mu}}^{T} \end{pmatrix}$$

- The permittivity and permeability dyadics satisfy $\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}^T = \overline{\overline{\epsilon}}^*$, $\overline{\overline{\mu}} = \overline{\overline{\mu}}^T = \overline{\overline{\mu}}^*$ or $\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ are real and symmetric dyadics

- The magnetoelectric dyadics satisfy $\overline{\overline{\xi}} = -\overline{\overline{\xi}}^* = -\overline{\overline{\zeta}}$ and $\overline{\overline{\zeta}} = -\overline{\overline{\zeta}}^* = -\overline{\overline{\xi}}$, whence $\overline{\overline{\xi}}$ and $\overline{\overline{\zeta}}$ are imaginary dyadics satisfying $\overline{\overline{\xi}}^T = -\overline{\overline{\zeta}}$.

- Writing $\overline{\overline{\zeta}} = \overline{\overline{\chi}} + j\overline{\overline{\kappa}}$, $\overline{\overline{\xi}} = \overline{\overline{\chi}} - j\overline{\overline{\kappa}}$, $\overline{\overline{\kappa}}$ must be real and symmetric and $\overline{\overline{\chi}}$ imaginary and antisymmetric.

Problems

4.1 Derive the condition for the imaginary parts of the medium parameters of a lossy bi-isotropic chiral medium:

$$\kappa_{im}^2 < \frac{\mu_{im}\epsilon_{im}}{\mu_o\epsilon_o}$$

by requiring that the condition $\Re\{\nabla\cdot\mathbf{S}\} < 0$ be valid for all possible fields in the chiral medium.

4.2 Consider a plane wave in a lossless non-reciprocal bi-isotropic medium with real nonzero magnetoelectric parameters $\xi = \zeta \neq \pm\sqrt{\mu\epsilon}$,

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_o e^{-jkz}, \quad \mathbf{H}(z) = \mathbf{H}_o e^{-jkz}.$$

Show that this is a TEM wave and solve the factor k. Also show that linearly polarized electric and magnetic field vectors are not orthogonal for $\xi \neq 0$.

S-96.510 Advanced Field Theory
05. Field Transformations
Duality transformation

I.V.Lindell

Field transformations

- Electromagnetic transformations make it possible to find solutions to new problems in terms of old problems with known solutions. The solution process (often tedious) can be avoided:

$$\text{Problem} \;\rightarrow\; \text{Solution process} \;\rightarrow\; \text{Result}$$

$$\Downarrow$$

$$\text{Problem} \;\rightarrow\; \text{Transf.} \;\rightarrow\; \text{Known solution} \;\rightarrow\; \text{Inverse transf.} \;\rightarrow\; \text{Result}$$

- Duality transformation: linear transformation of fields
  induces transformation of sources and media

- Affine transformation: linear transformation of space
  induces transformation of fields, sources and media

Simple duality (Duality substitution)

- Symmetry in Maxwell equations (e.g., isotropic media):

$$\text{M1}: \quad \nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{M}$$

$$\text{M2}: \quad -\nabla \times \mathbf{E} = j\omega\mu\mathbf{H} + \mathbf{J}$$

- Replace

$$\mathbf{E} \to \mathbf{H}, \quad \mathbf{H} \to \mathbf{E}, \quad \epsilon \to -\mu, \quad \mu \to -\epsilon, \quad \mathbf{M} \to -\mathbf{J}, \quad \mathbf{J} \to -\mathbf{M}$$

- Maxwell equations invariant: M1→M2, M2→M1
- Not a transformation, dimensions are changed
- Can be used for transforming formulas

Example: transformation of formulas

- Electric field radiated by electric current

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')dV'$$

- Duality substitution:

$$\mathbf{E} \to \mathbf{H}, \quad \mathbf{H} \to \mathbf{E}, \quad \epsilon \to -\mu, \quad \mu \to -\epsilon, \quad \mathbf{M} \to -\mathbf{J}, \quad \mathbf{J} \to -\mathbf{M}$$

- Green dyadic $\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')$ depends on $k = \omega\sqrt{\mu\epsilon}$, which is invariant
- Magnetic field from magnetic current

$$\mathbf{H}(\mathbf{r}) = -j\omega\epsilon \int \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')dV'$$

- Electric field obtained through $\nabla\times$

$$\mathbf{E}(\mathbf{r}) = \frac{\nabla \times \mathbf{H}(\mathbf{r})}{j\omega\epsilon} = -\nabla \times \int \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')dV'$$

Classical duality

- Define duality transformation $\mathbf{E} \to \mathbf{E}_d$ etc (B scalar constant):

$$\mathbf{E}_d = B\mathbf{H}, \quad \mathbf{B}_d = -B\mathbf{D}, \quad \mathbf{M}_d = -B\mathbf{J}$$

- Transforms one Maxwell equation to another:

$$-\nabla \times \mathbf{E}_d = j\omega\mathbf{B}_d + \mathbf{M}_d \quad \Leftrightarrow \quad \nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J}$$

- Similarly

$$\mathbf{H}_d = C\mathbf{E}, \quad \mathbf{D}_d = -C\mathbf{B}, \quad \mathbf{J}_d = -C\mathbf{M}$$

$$\nabla \times \mathbf{H}_d = j\omega\mathbf{D}_d + \mathbf{J}_d \quad \Leftrightarrow \quad -\nabla \times \mathbf{E} = j\omega\mathbf{B} + \mathbf{M}$$

- Works for arbitrary scalars $B, C \neq 0$ (with correct dimensions)

Medium transformation

- Transformation for medium dyadics

$$\begin{pmatrix} \mathbf{D}_d \\ \mathbf{B}_d \end{pmatrix} = -\begin{pmatrix} C\overline{\overline{\mu}}\cdot\mathbf{H} + C\overline{\overline{\zeta}}\cdot\mathbf{E} \\ B\overline{\overline{\epsilon}}\cdot\mathbf{E} + B\overline{\overline{\xi}}\cdot\mathbf{H} \end{pmatrix} = -\begin{pmatrix} \overline{\overline{\mu}}C/B & \overline{\overline{\zeta}} \\ \overline{\overline{\xi}} & \overline{\overline{\epsilon}}B/C \end{pmatrix}\begin{pmatrix} \mathbf{E}_d \\ \mathbf{H}_d \end{pmatrix}$$

- Transformed medium dyadics

$$\begin{pmatrix} \overline{\overline{\epsilon}}_d & \overline{\overline{\xi}}_d \\ \overline{\overline{\zeta}}_d & \overline{\overline{\mu}}_d \end{pmatrix} = \begin{pmatrix} -\overline{\overline{\mu}}C/B & -\overline{\overline{\zeta}} \\ -\overline{\overline{\xi}} & -\overline{\overline{\epsilon}}B/C \end{pmatrix}$$

- Dual fields and sources exist in dual medium!
- Example: dual of isotropic medium = another isotropic medium:

$$\epsilon_d = -\frac{C}{B}\mu, \qquad \mu_d = -\frac{B}{C}\epsilon$$

Choice of parameters

- Duality transformation works for any two parameters $B, C \neq 0$:

$$\begin{pmatrix} \mathbf{E}_d \\ \mathbf{H}_d \end{pmatrix} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix},$$

- Usually B, C chosen to satisfy two conditions:
- (1) Duality transformation in an involution (equals its inverse)

$$\begin{pmatrix} \mathbf{E}_d \\ \mathbf{H}_d \end{pmatrix}_d = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}_d = BC \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \quad \Rightarrow \quad BC = 1,$$

- (2) Free-space medium $\overline{\overline{\epsilon}} = \epsilon_o \overline{\overline{I}}$, $\overline{\overline{\mu}} = \mu_o \overline{\overline{I}}$ and $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$ is self dual:

$$\overline{\overline{\epsilon}}_d = \overline{\overline{\epsilon}} = -\overline{\overline{\mu}} C/B \quad \text{and} \quad \overline{\overline{\mu}}_d = \overline{\overline{\mu}} = -\overline{\overline{\epsilon}} B/C$$

$$\epsilon_o = -\mu_o C/B, \quad \Rightarrow \quad B/C = -\mu_o/\epsilon_o = -\eta_o^2$$

- Two possible solutions: $B_\pm = 1/C_\pm = \mp j\eta_o = \mp j\sqrt{\mu_o/\epsilon_o}$

Two duality transformations

- Right-hand transformation (+) and left-hand transformation (-) [explained later]

$$\begin{pmatrix} \mathbf{E}_{d\pm} \\ \mathbf{H}_{d\pm} \end{pmatrix} = \pm j \begin{pmatrix} 0 & -\eta_o \\ 1/\eta_o & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

- Medium transformation rules same for both cases: $\overline{\overline{\epsilon}}_{d\pm} = \overline{\overline{\epsilon}}_d$ etc

$$\begin{pmatrix} \overline{\overline{\epsilon}}_d & \overline{\overline{\xi}}_d \\ \overline{\overline{\zeta}}_d & \overline{\overline{\mu}}_d \end{pmatrix} = \begin{pmatrix} \overline{\overline{\mu}}/\eta_o^2 & -\overline{\overline{\zeta}} \\ -\overline{\overline{\xi}} & \overline{\overline{\epsilon}}\eta_o^2 \end{pmatrix} = \begin{pmatrix} \epsilon_o(\overline{\overline{\mu}}/\mu_o) & -\overline{\overline{\zeta}} \\ -\overline{\overline{\xi}} & \mu_o(\overline{\overline{\epsilon}}/\epsilon_o) \end{pmatrix}$$

- Relative dual permittivity and permeability: $\overline{\overline{\epsilon}}_{rd} = \overline{\overline{\mu}}_r, \quad \overline{\overline{\mu}}_{rd} = \overline{\overline{\epsilon}}_r$
- Isotropic medium: wave number: $k_d = \omega\sqrt{\mu_d\epsilon_d} = \omega\sqrt{\mu_o\epsilon_r\epsilon_o\mu_r} = k$
- Wave impedance $\eta_d = \sqrt{\mu_d/\epsilon_d} = \sqrt{\mu_o\epsilon_r/\epsilon_o\mu_r} = \eta_o^2/\eta$

Example

- Incident plane-wave $\mathbf{E}^i = \mathbf{u}Ae^{-j\mathbf{k}\cdot\mathbf{r}}, \mathbf{H}^i = \mathbf{v}Be^{-j\mathbf{k}\cdot\mathbf{r}}$ scattering from a dielectric object ($\epsilon_r = \alpha, \mu_r = 1$) $\Rightarrow$ scattered fields $\mathbf{E}^s, \mathbf{H}^s$
- Duality transformation (e.g., right-hand d_+) $\Rightarrow$ dual plane-wave

$$\mathbf{E}^i_{d_+} = -j\eta_o\mathbf{H}^i = -j\eta_o\mathbf{v}Be^{-j\mathbf{k}\cdot\mathbf{r}},$$

$$\mathbf{H}^i_{d_+} = -\mathbf{E}^i/j\eta_o = -\mathbf{u}(A/j\eta_o)e^{-j\mathbf{k}\cdot\mathbf{r}}$$

- Scattering from a dual object ($\mu_{rd} = \alpha,\ \epsilon_{rd} = 1$)
- Scattered fields $\mathbf{E}^s_{d_+} = -j\eta_o\mathbf{H}^s$ and $\mathbf{H}^s_{d+} = -\mathbf{E}^s/j\eta_o$ are dual to the original scattered fields
- If incident field self dual $\Rightarrow$ same incident field in dual problem

Self-dual quantities

- Self-dual fields invariant in right/left-hand duality transformation
- Self-dual fields in right-hand transformation

$$\mathbf{E}_+ = (\mathbf{E}_+)_{d_+} = -j\eta_o \mathbf{H}_+, \qquad \mathbf{H}_+ = (\mathbf{H}_+)_{d_+} = -\mathbf{E}_+/j\eta_o$$

- Self-dual fields in left-hand transformation

$$\mathbf{E}_- = (\mathbf{E}_-)_{d_-} = j\eta_o \mathbf{H}_-, \qquad \mathbf{H}_- = (\mathbf{H}_-)_{d_-} = \mathbf{E}_-/j\eta_o$$

- Self-dual decomposition of any field:

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-, \quad \mathbf{E}_\pm = \frac{1}{2}(\mathbf{E} \mp j\eta_o \mathbf{H}),$$

$$\mathbf{H} = \mathbf{H}_+ + \mathbf{H}_-, \quad \mathbf{H}_\pm = \frac{1}{2}(\mathbf{H} \mp \frac{1}{j\eta_o}\mathbf{E})$$

- Decomposition of sources: $\mathbf{J}_\pm = (\mathbf{J} \pm \mathbf{M}/j\eta_o)/2$

Self-dual plane wave

- Plane wave in air $\mathbf{E} = \mathbf{E}_o e^{-jk\mathbf{u}\cdot\mathbf{r}}$, $\mathbf{H} = \frac{1}{\eta_o}\mathbf{u}\times\mathbf{E}_o e^{-jk\mathbf{u}\cdot\mathbf{r}}$ (real $\mathbf{u}$)
- Self dual in right-hand transformation:

$$\mathbf{E} = \mathbf{E}_{d_+} \quad \Rightarrow \quad \mathbf{E}_o = -j\eta_o\mathbf{H}_o = -j\mathbf{u}\times\mathbf{E}_o$$

- Satisfies $\mathbf{u}\cdot\mathbf{E}_o = 0$ and $\mathbf{E}_o\cdot\mathbf{E}_o = 0$ (circular polarization)

$$\mathbf{p}(\mathbf{E}_o) = \frac{\mathbf{E}_o\times\mathbf{E}_o^*}{j\mathbf{E}_o\cdot\mathbf{E}_o^*} = \frac{-j(\mathbf{u}\times\mathbf{E}_o)\times\mathbf{E}_o^*}{j|\mathbf{E}_o|^2} = \mathbf{u}$$

- Right-hand circularly polarized plane wave is self dual in right-hand transformation and anti-self-dual in left-hand transformation.
- Decomposition $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$,
- Transformations $\mathbf{E}_{d_+} = \mathbf{E}_+ - \mathbf{E}_-$, $\quad \mathbf{E}_{d_-} = -\mathbf{E}_+ + \mathbf{E}_-$

Surface impedance condition

- Surface impedance condition ($\overline{\overline{Z}}_s = $ 2D dyadic)

$$\mathbf{E}_t = \overline{\overline{Z}}_s \cdot (\mathbf{n} \times \mathbf{H})$$

- Applying 2D inverse of a 2D dyadic $\overline{\overline{A}}$

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^T {}^\times_\times \mathbf{nn}}{\mathrm{spm}\overline{\overline{A}}}, \qquad \overline{\overline{A}} \cdot \overline{\overline{A}}^{-1} = \overline{\overline{A}}^{-1} \cdot \overline{\overline{A}} = \overline{\overline{I}}_t$$

$$\mathbf{n} \times \mathbf{H} = \overline{\overline{Z}}_s^{-1} \cdot \mathbf{E}_t = \frac{\overline{\overline{Z}}_s^T {}^\times_\times \mathbf{nn}}{\mathrm{spm}\overline{\overline{Z}}_s} \cdot \mathbf{E}_t = -\mathbf{n} \times \frac{\overline{\overline{Z}}_s^T}{\mathrm{spm}\overline{\overline{Z}}_s} \cdot (\mathbf{n} \times \mathbf{E})$$

- $\Rightarrow$ Another form of the same impedance condition:

$$\mathbf{H}_t = -\frac{\overline{\overline{Z}}_s^T}{\mathrm{spm}\overline{\overline{Z}}_s} \cdot (\mathbf{n} \times \mathbf{E})$$

Dual surface impedance

- Duality transformations of $\mathbf{E}_t = \overline{\overline{Z}}_s \cdot (\mathbf{n} \times \mathbf{H})$:

$$\mathbf{E}_{d_\pm t} = \overline{\overline{Z}}_{sd_\pm} \cdot (\mathbf{n} \times \mathbf{H}_{d_\pm})$$

$$\Rightarrow \quad \mp j\eta_o \mathbf{H}_t = \mp \frac{1}{j\eta_o} \overline{\overline{Z}}_{sd_\pm} \cdot (\mathbf{n} \times \mathbf{E}) = \pm j\eta_o \frac{\overline{\overline{Z}}_s^T}{\mathrm{spm}\overline{\overline{Z}}_s} \cdot (\mathbf{n} \times \mathbf{E})$$

- Identifying the dual of the surface impedance dyadic $\overline{\overline{Z}}_{sd_\pm} = \overline{\overline{Z}}_{sd}$ (same for both transformations)

$$\overline{\overline{Z}}_{sd} = \eta_o^2 \frac{\overline{\overline{Z}}_s^T}{\mathrm{spm}\overline{\overline{Z}}_s}$$

- Another form follows from $\mathrm{spm}(\overline{\overline{Z}}_s {}^\times_\times \mathbf{nn}) = \mathrm{spm}\overline{\overline{Z}}_s$

$$\overline{\overline{Z}}_{sd} = \eta_o^2 (\overline{\overline{Z}}_s {}^\times_\times \mathbf{nn})^{-1}$$

Self-dual surface impedance

- Self-dual condition:

$$\overline{\overline{Z}}_s = \overline{\overline{Z}}_{sd} = \left(\frac{\eta_o^2}{\mathrm{spm}\overline{\overline{Z}}_s}\right)\overline{\overline{Z}}_s^T \quad \Rightarrow \quad \mathrm{spm}\overline{\overline{Z}}_s = \frac{\eta_o^4}{\mathrm{spm}\overline{\overline{Z}}_s}$$

- Two conditions: $\mathrm{spm}\overline{\overline{Z}}_s = \pm\eta_o^2 \;\Rightarrow\; \overline{\overline{Z}}_s^T = \pm\overline{\overline{Z}}_s$
- Antisymmetric (nonreciprocal) solution $\overline{\overline{Z}}_s = \pm j\eta_o\mathbf{n}\times\overline{\overline{I}}$
- Symmetric (reciprocal) surface impedance in terms of ONB $(\mathbf{v}, \mathbf{w}, \mathbf{n})$

$$\overline{\overline{Z}}_s = Z_v\mathbf{vv} + Z_w\mathbf{ww}, \;\; \Rightarrow \;\; \mathrm{spm}\overline{\overline{Z}}_s = Z_vZ_w = \eta_o^2$$

- Self-dual reciprocal impedance surface has the form

$$\overline{\overline{Z}}_s = \eta_o(\alpha\mathbf{vv} + \alpha^{-1}\mathbf{ww})$$

- Limiting cases $\alpha = 0, \infty$: soft and hard (tuned corrugated) surface

Self-dual media

- Free space = self dual medium. Other possibilities?

$$\begin{pmatrix} \overline{\overline{\epsilon}}_d & \overline{\overline{\xi}}_d \\ \overline{\overline{\zeta}}_d & \overline{\overline{\mu}}_d \end{pmatrix} = \begin{pmatrix} \overline{\overline{\mu}}/\eta_o^2 & -\overline{\overline{\zeta}} \\ -\overline{\overline{\xi}} & \overline{\overline{\epsilon}}\eta_o^2 \end{pmatrix} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}$$

- Denote $\overline{\overline{\xi}} = (\overline{\overline{\chi}} - j\overline{\overline{\kappa}})\sqrt{\mu_o\epsilon_o}, \quad \overline{\overline{\zeta}} = (\overline{\overline{\chi}} + j\overline{\overline{\kappa}})\sqrt{\mu_o\epsilon_o}$

$$\overline{\overline{\epsilon}}_r = \overline{\overline{\mu}}_r = \overline{\overline{\alpha}}, \quad \overline{\overline{\xi}} = -\overline{\overline{\zeta}} \quad \Rightarrow \quad \overline{\overline{\chi}} = 0,$$

- Self-dual medium can be defined in terms of two arbitrary dyadics:

$$\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\alpha}}\epsilon_o & -j\overline{\overline{\kappa}}\sqrt{\mu_o\epsilon_o} \\ j\overline{\overline{\kappa}}\sqrt{\mu_o\epsilon_o} & \overline{\overline{\alpha}}\mu_o \end{pmatrix}$$

- More general duality transformation ⇒ more general self-dual medium

General duality transformation

- General duality transformation T for the fields (six-vector form)

$$\mathsf{e}_d = T\mathsf{e}, \quad \mathsf{e} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad AD - BC \neq 0,$$

- Transformed Maxwell equations can be expanded as

$$J\nabla \times \mathsf{e}_d = j\omega \mathsf{M}_d \cdot \mathsf{e}_d + \mathsf{j}_d, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$JT\nabla \times \mathsf{e} = [JTJ^{-1}]J\nabla \times \mathsf{e} = j\omega \mathsf{M}_d T \cdot \mathsf{e} + \mathsf{j}_d$$

$$J\nabla \times \mathsf{e} = j\omega [JT^{-1}J^{-1}]\mathsf{M}_d T \cdot \mathsf{e} + [JT^{-1}J^{-1}]\mathsf{j}_d$$

- Identify dual sources from $\mathsf{j} = [JT^{-1}J^{-1}]\mathsf{j}_d$:

$$\begin{pmatrix} \mathbf{J}_d \\ \mathbf{M}_d \end{pmatrix} = JTJ^{-1}\mathsf{j} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \mathbf{J} \\ \mathbf{M} \end{pmatrix}$$

Duality transformation of media

- Identify dual medium six-dyadic:

$$\mathsf{M} = [JT^{-1}J^{-1}\mathsf{M}_d T], \quad \Rightarrow \quad \mathsf{M}_d = JTJ^{-1}\mathsf{M}T^{-1}$$

$$\begin{pmatrix} \overline{\overline{\epsilon}}_d \\ \overline{\overline{\xi}}_d \\ \zeta_d \\ \overline{\overline{\mu}}_d \end{pmatrix} = \frac{1}{AD - BC} \begin{pmatrix} D^2 & -CD & -CD & C^2 \\ -BD & AD & BC & -AC \\ -BD & BC & AD & -AC \\ B^2 & -AB & -AB & A^2 \end{pmatrix} \begin{pmatrix} \overline{\overline{\epsilon}} \\ \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} \\ \overline{\overline{\mu}} \end{pmatrix}$$

- Dual medium dyadics linear combinations of original medium dyadics
- Self-dual medium with respect to a transformation (A, B, C, D):

$$\overline{\overline{\epsilon}}_d = \overline{\overline{\epsilon}}, \quad \overline{\overline{\xi}}_d = \overline{\overline{\xi}}, \quad \overline{\overline{\zeta}}_d = \overline{\overline{\zeta}}, \quad \overline{\overline{\mu}}_d = \overline{\overline{\mu}}$$

- Four conditions reduce to only two

Self-dual media

- Self-dual medium conditions for general A, B, C, D reduce to

$$(A-D)\overline{\overline{\epsilon}} + C(\overline{\overline{\xi}} + \overline{\overline{\zeta}}) = 0, \qquad (D-A)\overline{\overline{\mu}} + B(\overline{\overline{\xi}} + \overline{\overline{\zeta}}) = 0$$

- Define $\overline{\overline{\xi}}, \overline{\overline{\zeta}} = (\overline{\overline{\chi}}_r \mp j\overline{\overline{\kappa}}_r)\sqrt{\mu\epsilon}, \quad \eta = \sqrt{\mu/\epsilon}$
- $\Rightarrow$ $\overline{\overline{\epsilon}}$, $\overline{\overline{\mu}}$ and $\overline{\overline{\chi}}$ must be multiples of the same dyadic, say, $\overline{\overline{\alpha}}$ (Previously $A = D = 0$, whence $\overline{\overline{\chi}} = 0$)
- $\overline{\overline{\xi}} - \overline{\overline{\zeta}} = -2j\overline{\overline{\kappa}}_r\sqrt{\mu\epsilon}$ is always invariant: $\overline{\overline{\kappa}}_{rd} = \overline{\overline{\kappa}}_r$
- General self-dual medium: medium dyadics have the form

$$\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \epsilon & \chi_r\sqrt{\mu\epsilon} \\ \chi_r\sqrt{\mu\epsilon} & \mu \end{pmatrix} \overline{\overline{\alpha}} + \begin{pmatrix} 0 & -j\sqrt{\mu\epsilon} \\ j\sqrt{\mu\epsilon} & 0 \end{pmatrix} \overline{\overline{\kappa}}_r$$

Special case: involutory duality

- Require: $T^{-1} = T, \Rightarrow T^2 = I \quad \Rightarrow A = -D = \pm\sqrt{1 - BC}$
- Assuming self-dual medium with parameters $\epsilon, \mu, \chi_r = \sin\theta$, ($\overline{\overline{\alpha}}, \overline{\overline{\kappa}}_r$ do not matter), transformation matrix T can be solved:

$$D = -A = \pm j\chi_r/\sqrt{1 - \chi_r^2} = \pm j \tan\theta$$

$$B/\eta = -\eta C = A/\chi_r = \mp j/\cos\theta$$

$$T_\pm = \pm j \begin{pmatrix} -\tan\theta & -\eta/\cos\theta \\ 1/\eta\cos\theta & \tan\theta \end{pmatrix}, (\eta = \eta_o, \theta = 0 \;\Rightarrow\; \text{classical duality})$$

- Self-dual fields: $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-, \quad \mathbf{H} = \mathbf{H}_+ + \mathbf{H}_-$

$$T_\pm \begin{pmatrix} \mathbf{E}_\pm \\ \mathbf{H}_\pm \end{pmatrix} = \begin{pmatrix} \mathbf{E}_\pm \\ \mathbf{H}_\pm \end{pmatrix}$$

- Relation $\mathbf{E}_\pm = Z_\pm \mathbf{H}_\pm$, wave impedances $Z_\pm = \mp j\eta e^{\mp j\theta}$

Self-dual (Bohren) decomposition 1

- Field decomposition in self-dual parts $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-,\ \mathbf{H} = \mathbf{H}_+ + \mathbf{H}_-$

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -e^{j\theta}/j\eta & e^{-j\theta}/j\eta \end{pmatrix} \begin{pmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{pmatrix} = K \begin{pmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{pmatrix} = \frac{1}{2\cos\theta} \begin{pmatrix} e^{-j\theta} & -j\eta \\ e^{j\theta} & j\eta \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = K^{-1} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

- Maxwell equations are decoupled for homogeneous self-dual media:

$$\nabla \times \begin{pmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{pmatrix} = K^{-1} \nabla \times \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

$$= K^{-1} \begin{pmatrix} -j\omega\overline{\overline{\zeta}} & -j\omega\overline{\overline{\mu}} \\ j\omega\overline{\overline{\epsilon}} & j\omega\overline{\overline{\xi}} \end{pmatrix} K \begin{pmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{pmatrix} + K^{-1} \begin{pmatrix} -\mathbf{M} \\ \mathbf{J} \end{pmatrix}$$

$$= \begin{pmatrix} k\overline{\overline{\tau}}_+ & 0 \\ 0 & k\overline{\overline{\tau}}_- \end{pmatrix} \begin{pmatrix} \mathbf{E}_+ \\ \mathbf{E}_- \end{pmatrix} - \begin{pmatrix} \mathbf{M}_+ \\ \mathbf{M}_- \end{pmatrix}, \quad \overline{\overline{\tau}}_\pm = \overline{\overline{\kappa}}_r \pm \cos\theta \overline{\overline{\alpha}}$$

Self-dual (Bohren) decomposition 2

- Maxwell equations for self-dual fields

$$\nabla \times \mathbf{E}_\pm = \pm k \overline{\overline{\tau}}_\pm \cdot \mathbf{E}_\pm - \mathbf{M}_\pm$$

$$\mathbf{M}_\pm = \frac{1}{2\cos\theta}(e^{\mp j\theta}\mathbf{M} \pm j\eta\mathbf{J}) = \pm j\eta e^{\mp j\theta}\mathbf{J}_\pm$$

- They can also be expressed as four equations (twice the same)

$$-\nabla \times \mathbf{E}_\pm = j\omega\overline{\overline{\mu}}_\pm \cdot \mathbf{H}_\pm + \mathbf{M}_\pm, \qquad \nabla \times \mathbf{H}_\pm = j\omega\overline{\overline{\epsilon}}_\pm \cdot \mathbf{E}_\pm + \mathbf{J}_\pm$$

- Equivalent medium dyadics for the decomposed fields

$$\overline{\overline{\epsilon}}_\pm = \pm\frac{k}{j\omega Z_\pm}\overline{\overline{\tau}}_\pm = \epsilon e^{\pm j\theta}\overline{\overline{\tau}}_\pm, \qquad \overline{\overline{\mu}}_\pm = \mp\frac{k}{j\omega}\overline{\overline{\tau}}_\pm Z_\pm = \mu e^{\mp j\theta}\overline{\overline{\tau}}_\pm$$

- Problem in a self-dual bi-anisotropic medium splits in two parts each associated with an effective anisotropic medium.

Power propagation in self-dual medium

- Assume homogeneous and lossless self-dual medium: $\epsilon, \mu, \overline{\overline{\alpha}}, \theta, \overline{\overline{\kappa}}_r$ real
- Propagating power = real part of Poynting vector

$$\frac{1}{2}\Re\{\mathbf{E}\times\mathbf{H}^*\} = \frac{1}{2}\Re\{(\mathbf{E}_+ + \mathbf{E}_-)\times\big(\frac{\mathbf{E}_+}{Z_+} + \frac{\mathbf{E}_-}{Z_-}\big)^*\} =$$

$$\frac{1}{2\eta}\Re\{-j\mathbf{E}_+\times\mathbf{E}_+^* e^{-j\theta} + j\mathbf{E}_-\times\mathbf{E}_-^* e^{j\theta} + j[\mathbf{E}_+\times\mathbf{E}_-^* e^{j\theta} + \mathbf{E}_+^*\times\mathbf{E}_- e^{-j\theta}]\}$$

$$= \cos\theta[\mathbf{p}(\mathbf{E}_+)\frac{|\mathbf{E}_+|^2}{2\eta} - \mathbf{p}(\mathbf{E}_-)\frac{|\mathbf{E}_-|^2}{2\eta}], \qquad Z_\pm = \mp j\eta e^{\mp j\theta}$$

- Decomposed fields do not couple power to one another!
- $\mathbf{p}(\mathbf{E}_+)$ right-hand polarization, $-\mathbf{p}(\mathbf{E}_-)$ left-hand polarization

Problems

5.1 Find conditions for the bi-anisotropic medium which can be transformed to an anisotropic medium, i.e., satisfying $\overline{\overline{\xi}}_d = \overline{\overline{\zeta}}_d = 0$. Define the parameters A, B, C, D required for the transformation.

5.2 Study the reflection of a plane wave $\mathbf{E}_o e^{-jk_o z}$ propagating in air from a self-dual surface impedance $\overline{\overline{Z}}_s = \eta_o(\mathbf{u}_x\mathbf{u}_x\alpha + \mathbf{u}_y\mathbf{u}_y/\alpha)$ at $z = 0$. Find the two-dimensional reflection dyadic $\overline{\overline{R}}$ giving the reflected field as $\overline{\overline{R}} \cdot \mathbf{E}_o e^{jk_o z}$. Show that the handedness of the wave is not changed in the reflection.

S-96.510 Advanced Field Theory
6. Affine transformation

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Affine transformation

- Affine transformation = linear deformation of space (stretching, compressing, rotating, mirror-imaging etc.)
- Affine transformation makes microscopic distortion to medium. For example: isotropic medium may become anisotropic
- Affine transformation defined by complete dyadic $\overline{\overline{A}}$ ($\det\overline{\overline{A}} \neq 0$)

$$\text{space} \quad \mathbf{r} \rightarrow \mathbf{r}_a = \overline{\overline{A}} \cdot \mathbf{r} \quad \text{affine space}$$

- Property of nabla operator

$$\nabla\mathbf{r} = \nabla x \mathbf{u}_x + \nabla y \mathbf{u}_y + \nabla z \mathbf{u}_z = \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y + \mathbf{u}_z\mathbf{u}_z = \overline{\overline{I}}$$

- Nabla operator in affine space

$$\overline{\overline{I}} = \nabla_a \mathbf{r}_a = \nabla_a(\overline{\overline{A}} \cdot \mathbf{r}) = \overline{\overline{A}}^{-1T} \cdot \nabla\mathbf{r} \cdot \overline{\overline{A}}^{T}, \quad \Rightarrow \quad \nabla_a = \overline{\overline{A}}^{-1T} \cdot \nabla$$

Basic tools

- Applying six-vector representation

$$\mathsf{e} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad \mathsf{d} = \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}, \quad \mathsf{j} = \begin{pmatrix} \mathbf{J} \\ \mathbf{M} \end{pmatrix}, \quad \mathsf{M} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Original Maxwell equations

$$J\nabla \times \mathsf{e}(\mathbf{r}) = j\omega \mathsf{d}(\mathbf{r}) + \mathsf{j}(\mathbf{r}), \qquad \mathsf{d} = \mathsf{M} \cdot \mathsf{e}$$

- are affine-transformed to

$$J\nabla_a \times \mathsf{e}_a(\mathbf{r}_a) = j\omega \mathsf{d}_a(\mathbf{r}_a) + \mathsf{j}_a(\mathbf{r}_a), \qquad \mathsf{d}_a = \mathsf{M}_a \cdot \mathsf{e}_a$$

- Analytic tools needed: identity

$$(\overline{\overline{D}} \cdot \mathbf{a}) \times (\overline{\overline{D}} \cdot \mathbf{b}) = \overline{\overline{D}}^{(2)} \cdot (\mathbf{a} \times \mathbf{b}) = (\det\overline{\overline{D}})\overline{\overline{D}}^{-1T} \cdot (\mathbf{a} \times \mathbf{b})$$

- Dyadic relation $[\overline{\overline{D}}^{-1}]^{(2)} = \overline{\overline{D}}^{(-2)} = \overline{\overline{D}}^{T}/\det\overline{\overline{D}}$

Affine transformation of fields and sources

- Transforming Maxwell equations

$$(\overline{\overline{D}}\cdot\nabla)\times(\overline{\overline{D}}\cdot\mathsf{e}(\mathbf{r})) = \overline{\overline{D}}^{(2)}\cdot(\nabla\times\mathsf{e}(\mathbf{r})) = \overline{\overline{D}}^{(2)}\cdot[j\omega\mathsf{d}(\mathbf{r})+\mathsf{j}(\mathbf{r})]$$

- Because $\nabla_a = \overline{\overline{A}}^{-1T}\cdot\nabla$, choose $\overline{\overline{D}} = \overline{\overline{A}}^{-1T}$ which gives

$$J\nabla_a\times\mathsf{e}_a(\mathbf{r}_a) = j\omega\mathsf{d}_a(\mathbf{r}_a)+\mathsf{j}_a(\mathbf{r}_a)$$

- Identify term by term ⇒ transformed fields and sources:

$$\mathsf{e}_a(\mathbf{r}_a) = \overline{\overline{D}}\cdot\mathsf{e}(\mathbf{r}) = \overline{\overline{A}}^{-1T}\cdot\mathsf{e}(\overline{\overline{A}}^{-1}\cdot\mathbf{r}_a)$$

$$\mathsf{d}_a(\mathbf{r}_a) = \overline{\overline{D}}^{(2)}\cdot\mathsf{d}(\mathbf{r}) = \overline{\overline{A}}^{(-2)T}\cdot\mathsf{d}(\overline{\overline{A}}^{-1}\cdot\mathbf{r}_a) = \overline{\overline{A}}\cdot\mathsf{d}(\overline{\overline{A}}^{-1}\cdot\mathbf{r}_a)/\det\overline{\overline{A}}$$

$$\mathsf{j}_a(\mathbf{r}_a) = \overline{\overline{D}}^{(2)}\cdot\mathsf{j}(\mathbf{r}) = \overline{\overline{A}}^{(-2)T}\cdot\mathsf{j}(\overline{\overline{A}}^{-1}\cdot\mathbf{r}_a) = \overline{\overline{A}}\cdot\mathsf{j}(\overline{\overline{A}}^{-1}\cdot\mathbf{r}_a)/\det\overline{\overline{A}}$$

Affine transformation of media

- Medium equations

$$\mathsf{M}_a \cdot \mathsf{e}_a = \mathsf{d}_a = \frac{\overline{\overline{A}}}{\det\overline{\overline{A}}} \cdot \mathsf{d} = \frac{\overline{\overline{A}}}{\det\overline{\overline{A}}} \cdot \mathsf{M} \cdot \mathsf{e} = \frac{\overline{\overline{A}}}{\det\overline{\overline{A}}} \cdot \mathsf{M} \cdot \overline{\overline{A}}^T \cdot \mathsf{e}_a$$

- Identify the medium six-dyadic:

$$\mathsf{M}_a = \frac{\overline{\overline{A}} \cdot \mathsf{M} \cdot \overline{\overline{A}}^T}{\det\overline{\overline{A}}}$$

- Transformation rules for all medium dyadics are similar:

$$\overline{\overline{\epsilon}}_a = \frac{\overline{\overline{A}} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{A}}^T}{\det\overline{\overline{A}}}, \qquad \overline{\overline{\xi}}_a = \frac{\overline{\overline{A}} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{A}}^T}{\det\overline{\overline{A}}},$$

$$\overline{\overline{\zeta}}_a = \frac{\overline{\overline{A}} \cdot \overline{\overline{\zeta}} \cdot \overline{\overline{A}}^T}{\det\overline{\overline{A}}}, \qquad \overline{\overline{\mu}}_a = \frac{\overline{\overline{A}} \cdot \overline{\overline{\mu}} \cdot \overline{\overline{A}}^T}{\det\overline{\overline{A}}},$$

Rotation transformation

- Rotation transformation around an axis defined by $\mathbf{u}$

$$\overline{\overline{A}} = \overline{\overline{R}}(\theta) = \mathbf{uu} + \cos\theta\overline{\overline{I}}_t + \sin\theta\mathbf{u}\times\overline{\overline{I}} = e^{\theta\mathbf{u}\times\overline{\overline{I}}}, \quad \overline{\overline{R}}(0) = \overline{\overline{I}}$$

$$\overline{\overline{R}}^T(\theta) = \overline{\overline{R}}^{-1}(\theta) = \overline{\overline{R}}(-\theta), \quad \overline{\overline{R}}^{(2)}(\theta) = \overline{\overline{R}}(\theta), \quad \det\overline{\overline{R}}(\theta) = 1$$

- Field vectors and source vectors are transformed as

$$\mathbf{E}_a(\mathbf{r}_a) = \overline{\overline{R}}(\theta)\cdot\mathbf{E}(\overline{\overline{R}}(-\theta)\cdot\mathbf{r}), \quad \mathbf{J}_a(\mathbf{r}_a) = \overline{\overline{R}}(\theta)\cdot\mathbf{J}(\overline{\overline{R}}(-\theta)\cdot\mathbf{r})$$

- All medium parameter dyadics are transformed as

$$\overline{\overline{\epsilon}}_a = \overline{\overline{A}}\cdot\overline{\overline{\epsilon}}\cdot\overline{\overline{A}}^T/\det\overline{\overline{A}} = \overline{\overline{R}}(\theta)\cdot\overline{\overline{\epsilon}}\cdot\overline{\overline{R}}(-\theta) = \sum(\overline{\overline{R}}(\theta)\cdot\mathbf{a}_i)(\overline{\overline{R}}(\theta)\cdot\mathbf{b}_i)$$

- $\overline{\overline{R}}(\theta)\cdot\overline{\overline{I}}\cdot\overline{\overline{R}}(-\theta) = \overline{\overline{I}} \;\Rightarrow$ bi-isotropic media are invariant in rotation
- Rotation transformation is not very interesting! General transformation $\overline{\overline{A}} = \overline{\overline{S}}\cdot\overline{\overline{R}}$. Interest in symmetric transformations $\overline{\overline{A}} = \overline{\overline{S}}$.

Uniaxial transformation

- Uniaxial transformation: stretching in $\mathbf{u}_z$ and $\perp\mathbf{u}_z$ directions

$$\overline{\overline{S}} = \alpha\overline{\overline{I}}_t + \beta\mathbf{u}_z\mathbf{u}_z, \quad \overline{\overline{S}}^{-1} = \alpha^{-1}\overline{\overline{I}}_t + \beta^{-1}\mathbf{u}_z\mathbf{u}_z, \quad \det\overline{\overline{S}} = \alpha^2\beta$$

- Field vectors are transformed as

$$\mathbf{E}_a(\mathbf{r}_a) = \overline{\overline{S}}^{-1}\cdot\mathbf{E}(\overline{\overline{S}}^{-1}\cdot\mathbf{r}_a) = [\alpha^{-1}\mathbf{E}_t + \beta^{-1}\mathbf{u}_z E_z](\alpha^{-1}\boldsymbol{\rho} + \beta^{-1}\mathbf{u}_z z)$$

- Source vectors are transformed similarly

- All medium parameter dyadics are transformed as

$$\overline{\overline{\epsilon}}_a = \beta^{-1}\overline{\overline{I}}_t\cdot\overline{\overline{\epsilon}}\cdot\overline{\overline{I}}_t + \alpha^{-1}[\overline{\overline{I}}_t\cdot\overline{\overline{\epsilon}}\cdot\mathbf{u}_z\mathbf{u}_z + \mathbf{u}_z\mathbf{u}_z\cdot\overline{\overline{\epsilon}}\cdot\overline{\overline{I}}_t] + \alpha^{-2}\beta\mathbf{u}_z\mathbf{u}_z(\overline{\overline{\epsilon}}:\mathbf{u}_z\mathbf{u}_z)$$

- (Bi-)isotropic medium is transformed to uniaxially (bi-)anisotropic medium

Generalization: triaxial transformation

- Triaxial transformation: stretching in three orthogonal directions

$$\overline{\overline{S}} = \sum \alpha_i \mathbf{u}_i \mathbf{u}_i, \quad \overline{\overline{S}}^{-1} = \sum \alpha_i^{-1} \mathbf{u}_i \mathbf{u}_i, \quad \mathrm{det}\overline{\overline{S}} = \alpha_1 \alpha_2 \alpha_3$$

- Field and source vectors are transformed as

$$\mathbf{E}_a(\mathbf{r}_a) = \sum \alpha_i^{-1} \mathbf{u}_i \mathbf{u}_i \cdot \mathbf{E}(\sum \alpha_i^{-1} \mathbf{u}_i \mathbf{u}_i \cdot \mathbf{r})$$

- All medium parameter dyadics are transformed as

$$\overline{\overline{\epsilon}}_a = \sum \sum \frac{\alpha_i \alpha_j}{\alpha_1 \alpha_2 \alpha_3} \mathbf{u}_i \mathbf{u}_j (\mathbf{u}_i \mathbf{u}_j : \overline{\overline{\epsilon}})$$

- Bi-isotropic medium is transformed to a bi-anisotropic medium. All medium dyadics multiples of the same dyadic $\overline{\overline{S}}^2$.
- Conversely: if medium dyadics are multiples of the same symmetric dyadic, bi-anisotropic medium can be transformed to a bi-isotropic medium.

Use of affine transformation

- Affine transformation changes sources, fields, media and boundaries both geometrically and electrically
- Point source remains point source ⇒ field solutions (Green dyadics) can be obtained for transformed media
- Boundaries are changed: Real symmetric $\overline{\overline{A}}$: isotropic sphere ⇒ anisotropic ellipsoid
- Surface equation $f(\mathbf{r}) = 0$ becomes $f(\overline{\overline{A}}^{-1} \cdot \mathbf{r}_a) = f_a(\mathbf{r}_a) = 0$
- Normal $\mathbf{n} \sim \nabla f(\mathbf{r})$ and $\mathbf{E}, \mathbf{H}$ transform similarly: $\mathbf{n}_a = \overline{\overline{A}}^{-1T} \cdot \mathbf{n}$

$$\overline{\overline{Z}}_{sa} \cdot (\mathbf{n}_a \times \mathbf{H}_a) = \mathbf{E}_{at} = \overline{\overline{A}}^{-1T} \cdot \mathbf{E}_t = \overline{\overline{A}}^{-1T} \cdot \overline{\overline{Z}}_s \cdot (\mathbf{n} \times \mathbf{H}) = \overline{\overline{A}}^{-1T} \cdot \overline{\overline{Z}}_s \cdot [(\overline{\overline{A}}^T \cdot \mathbf{n}_a) \times (\overline{\overline{A}}^T \cdot \mathbf{H})]$$

- Surface impedance dyadic transforms as

$$\overline{\overline{Z}}_{sa} = \overline{\overline{A}}^{-1T} \cdot \overline{\overline{Z}}_s \cdot \overline{\overline{A}}^{(2)T} = \overline{\overline{A}}^{-1T} \cdot \overline{\overline{Z}}_s \cdot \overline{\overline{A}}^{-1} \det \overline{\overline{A}}$$

Medium transformations 1

- Isotropic medium → anisotropic medium (**affine-isotropic** medium)

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix} \overline{\overline{I}} \quad \to \quad \begin{pmatrix} \overline{\overline{\epsilon}}_a & 0 \\ 0 & \overline{\overline{\mu}}_a \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\overline{\overline{A}} \cdot \overline{\overline{A}}^T}{\mathrm{det}\overline{\overline{A}}}$$

- Dyadics $\overline{\overline{\epsilon}}_a$ and $\overline{\overline{\mu}}_a$ symmetric and related by condition $\mu\overline{\overline{\epsilon}}_a = \epsilon\overline{\overline{\mu}}_a$

- Conversely: anisotropic medium → isotropic medium $\epsilon_a\overline{\overline{I}}$, $\mu_a\overline{\overline{I}}$ possible only if $\overline{\overline{\epsilon}} = \epsilon\overline{\overline{S}}$, $\overline{\overline{\mu}} = \mu\overline{\overline{S}}$, $\overline{\overline{S}}$ symmetric. Transformation dyadic $\overline{\overline{A}} = \alpha\overline{\overline{S}}^{1/2}$ with $\epsilon\mu_a = \mu\epsilon_a$ and $\alpha = \epsilon_a/(\epsilon\sqrt{\mathrm{det}\overline{\overline{S}}}) = \mu_a/(\mu\sqrt{\mathrm{det}\overline{\overline{S}}})$

- Medium is **affine-bi-isotropic** if medium six-dyadic is the form

$$\mathsf{M} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} \overline{\overline{S}},$$

Medium transformations 2

- What kind of medium transformations are possible?
- Symmetric dyadics transform to symmetric dyadics, antisymmetric to antisymmetric
- Magnetic anisotropy to electric anisotropy when $\overline{\overline{\mu}}$ symmetric $\Rightarrow \overline{\overline{\epsilon}}_a$ symmetric

$$\begin{pmatrix} \epsilon\overline{\overline{I}} & 0 \\ 0 & \overline{\overline{\mu}} \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \overline{\overline{\epsilon}}_a & 0 \\ 0 & \mu_a\overline{\overline{I}} \end{pmatrix}$$

- Ferrite $\leftrightarrow$ magnetoplasma not possible!
- Non-bi-anisotropic ($\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$) $\leftrightarrow$ bi-anisotropic not possible!
- Combination with duality transformation gives more possibilities

Simple use of affine transformation

- Scalar Helmholtz equation with symmetric $\overline{\overline{S}}$

$$(\overline{\overline{S}} : \nabla\nabla + k^2) f(\mathbf{r}) = g(\mathbf{r})$$

- Define $\nabla = \overline{\overline{S}}^{-1/2} \cdot \nabla_a, \quad \Rightarrow \quad \mathbf{r}_a = \overline{\overline{S}}^{-1/2} \cdot \mathbf{r}$ equation simplified to

$$(\nabla_a^2 + k^2) f_a(\mathbf{r}_a) = g(\mathbf{r}_a), \qquad f_a(\mathbf{r}_a) = f(\overline{\overline{S}}^{1/2} \cdot \mathbf{r}_a), \qquad g_a(\mathbf{r}_a) = g(\overline{\overline{S}}^{1/2} \cdot \mathbf{r}_a)$$

- If solution function $f_a(\mathbf{r}_a)$ can be found

$$\Rightarrow \quad f(\mathbf{r}) = f_a(\overline{\overline{S}}^{-1/2} \cdot \mathbf{r})$$

- Square root of symmetric dyadic not unique: e.g., same eigenvectors, eigenvalues $= \pm\sqrt{\lambda_i}$. Boundary conditions (radiation conditions) require uniqueness.

Duality-affine transformation

- Affine transformation can be connected to duality transformation

$$\mathsf{e}(\mathbf{r}) \;\to \mathsf{e}_{da}(\mathbf{r}_a) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \overline{\overline{A}}^{-1T} \cdot \mathsf{e}(\overline{\overline{A}}^{-1} \cdot \mathbf{r}_a)$$

$$\mathbf{E}_{da}(\mathbf{r}_a) = A\overline{\overline{A}}^{-1T} \cdot \mathbf{E}(\overline{\overline{A}}^{-1} \cdot \mathbf{r}_a) + B\overline{\overline{A}}^{-1T} \cdot \mathbf{H}(\overline{\overline{A}}^{-1} \cdot \mathbf{r}_a)$$

$$\mathbf{H}_{da}(\mathbf{r}_a) = C\overline{\overline{A}}^{-1T} \cdot \mathbf{E}(\overline{\overline{A}}^{-1} \cdot \mathbf{r}_a) + D\overline{\overline{A}}^{-1T} \cdot \mathbf{H}(\overline{\overline{A}}^{-1} \cdot \mathbf{r}_a)$$

- The medium dyadics become

$$\begin{pmatrix} \overline{\overline{\epsilon}}_{da} \\ \overline{\overline{\xi}}_{da} \\ \overline{\overline{\zeta}}_{da} \\ \overline{\overline{\mu}}_{da} \end{pmatrix} = \frac{1}{AD-BC} \begin{pmatrix} D^2 & -CD & -CD & C^2 \\ -BD & AD & BC & -AC \\ -BD & BC & AD & -AC \\ B^2 & -AB & -AB & A^2 \end{pmatrix} \frac{\overline{\overline{A}}}{\det\overline{\overline{A}}} \cdot \begin{pmatrix} \overline{\overline{\epsilon}} \\ \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} \\ \overline{\overline{\mu}} \end{pmatrix} \cdot \overline{\overline{A}}^{T}$$

- Medium dyadics $\overline{\overline{\epsilon}}_{da} \cdots$ linear combinations of $\overline{\overline{\epsilon}}_a, \cdots$

Reflection transformation

- Reflection in plane normal to $\mathbf{n}$

$$\overline{\overline{C}} = \overline{\overline{I}} - 2\mathbf{nn}, \qquad \overline{\overline{C}} \cdot \mathbf{a} = \overline{\overline{I}}_t \cdot \mathbf{a} - \mathbf{n}(\mathbf{n} \cdot \mathbf{a})$$

- Properties

$$\overline{\overline{C}}^2 = \overline{\overline{I}}, \quad \overline{\overline{C}}^T = \overline{\overline{C}}^{-1} = \overline{\overline{C}}, \qquad \mathrm{det}\overline{\overline{C}} = -1$$

- Reflection-transformed medium dyadics

$$\mathsf{M}_a = \frac{1}{\mathrm{det}\overline{\overline{C}}} \overline{\overline{C}} \cdot \mathsf{M} \cdot \overline{\overline{C}}^T = -\overline{\overline{C}} \cdot \mathsf{M} \cdot \overline{\overline{C}}$$

- Media invariant in reflection are strange. For example isotropic medium is not invariant: $\mathsf{M}_a = -\mathsf{M}$

- Can be combined with duality transformation!

Duality-reflection transformations

- Special duality transformation with $B = C = 0$ can be combined with reflection:

$$\begin{pmatrix} \overline{\overline{\epsilon}}_{da} \\ \overline{\overline{\xi}}_{da} \\ \zeta_{da} \\ \overline{\overline{\mu}}_{da} \end{pmatrix} = \frac{\overline{\overline{C}}}{\det\overline{\overline{C}}} \cdot \begin{pmatrix} (D/A)\overline{\overline{\epsilon}} \\ \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} \\ (A/D)\overline{\overline{\mu}} \end{pmatrix} \cdot \overline{\overline{C}}^T = \overline{\overline{C}} \cdot \begin{pmatrix} -(D/A)\overline{\overline{\epsilon}} \\ -\overline{\overline{\xi}} \\ -\overline{\overline{\zeta}} \\ -(A/D)\overline{\overline{\mu}} \end{pmatrix} \cdot \overline{\overline{C}}$$

- Isotropic medium invariant for $\Rightarrow D/A = -1$.
  Involution required $\Rightarrow$ two possibilities $D = -A = \pm 1$

- Define two transformations, $\mathbf{r}_c = \overline{\overline{C}} \cdot \mathbf{r}$

$$\mathsf{e}_c(\mathbf{r}_c) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix} \overline{\overline{C}}\cdot\mathsf{e}(\overline{\overline{C}}\cdot\mathbf{r}_c), \quad \mathsf{j}_c(\mathbf{r}_c) = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mp 1 \end{pmatrix} \overline{\overline{C}}\cdot\mathsf{j}(\overline{\overline{C}}\cdot\mathbf{r}_c)$$

- = Electric and magnetic reflection transformations.

Electric and magnetic reflection

- Two reflection transformations with $\mathbf{r}_c$ denoted by $\mathbf{r}$:
- Electric reflection

$$\begin{pmatrix} \mathbf{E}_c(\mathbf{r}) \\ \mathbf{H}_c(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \overline{\overline{C}}\cdot\mathbf{E}(\overline{\overline{C}}\cdot\mathbf{r}) \\ -\overline{\overline{C}}\cdot\mathbf{H}(\overline{\overline{C}}\cdot\mathbf{r}) \end{pmatrix}, \quad \begin{pmatrix} \mathbf{J}_c(\mathbf{r}) \\ \mathbf{M}_c(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \overline{\overline{C}}\cdot\mathbf{J}(\overline{\overline{C}}\cdot\mathbf{r}) \\ -\overline{\overline{C}}\cdot\mathbf{M}(\overline{\overline{C}}\cdot\mathbf{r}) \end{pmatrix}$$

- Magnetic reflection

$$\begin{pmatrix} \mathbf{E}_c(\mathbf{r}) \\ \mathbf{H}_c(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} -\overline{\overline{C}}\cdot\mathbf{E}(\overline{\overline{C}}\cdot\mathbf{r}) \\ \overline{\overline{C}}\cdot\mathbf{H}(\overline{\overline{C}}\cdot\mathbf{r}) \end{pmatrix}, \quad \begin{pmatrix} \mathbf{J}_c(\mathbf{r}) \\ \mathbf{M}_c(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} -\overline{\overline{C}}\cdot\mathbf{J}(\overline{\overline{C}}\cdot\mathbf{r}) \\ \overline{\overline{C}}\cdot\mathbf{M}(\overline{\overline{C}}\cdot\mathbf{r}) \end{pmatrix}$$

- Tangential fields at $\mathbf{n}\cdot\mathbf{r} = 0$ satisfying $\overline{\overline{C}}\cdot\mathbf{r} = \overline{\overline{C}}\cdot\boldsymbol{\rho} = \boldsymbol{\rho}$:

$$\mathbf{E}_{ct}(\boldsymbol{\rho}) = \mathbf{E}_t(\boldsymbol{\rho}), \quad \mathbf{H}_{ct}(\boldsymbol{\rho}) = -\mathbf{H}_t(\boldsymbol{\rho}), \quad \text{ER}$$

$$\mathbf{E}_{ct}(\boldsymbol{\rho}) = -\mathbf{E}_t(\boldsymbol{\rho}), \quad \mathbf{H}_{ct}(\boldsymbol{\rho}) = \mathbf{H}_t(\boldsymbol{\rho}), \quad \text{MR}$$

- Normal components with opposite signs

Summed fields in isotropic media

- Signs for electric/magnetic reflection transformation

$$\begin{pmatrix} \mathbf{E}(\mathbf{r}) + \mathbf{E}_c(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) + \mathbf{H}_c(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \mathbf{E}(\mathbf{r}) \pm \overline{\overline{C}} \cdot \mathbf{E}(\overline{\overline{C}} \cdot \mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \mp \overline{\overline{C}} \cdot \mathbf{H}(\overline{\overline{C}} \cdot \mathbf{r}) \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{J}(\mathbf{r}) + \mathbf{J}_c(\mathbf{r}) \\ \mathbf{M}(\mathbf{r}) + \mathbf{M}_c(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \mathbf{J}(\mathbf{r}) \pm \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r}) \\ \mathbf{M}(\mathbf{r}) \mp \overline{\overline{C}} \cdot \mathbf{M}(\overline{\overline{C}} \cdot \mathbf{r}) \end{pmatrix}$$

- Conditions on plane $\mathbf{n} \cdot \mathbf{r} = 0$ respectively

$$\mathbf{E}(\boldsymbol{\rho}) + \mathbf{E}_c(\boldsymbol{\rho}) = (\overline{\overline{I}} \pm \overline{\overline{C}}) \cdot \mathbf{E}(\boldsymbol{\rho}) = 2 \begin{pmatrix} \overline{\overline{I}}_t \\ \mathbf{nn} \end{pmatrix} \cdot \mathbf{E}(\boldsymbol{\rho}), \quad \begin{pmatrix} \text{PMC} \\ \text{PEC} \end{pmatrix} \text{condition}$$

$$\mathbf{H}(\boldsymbol{\rho}) + \mathbf{H}_c(\boldsymbol{\rho}) = (\overline{\overline{I}} \mp \overline{\overline{C}}) \cdot \mathbf{H}(\boldsymbol{\rho}) = 2 \begin{pmatrix} \mathbf{nn} \\ \overline{\overline{I}}_t \end{pmatrix} \cdot \mathbf{H}(\boldsymbol{\rho}), \quad \begin{pmatrix} \text{PMC} \\ \text{PEC} \end{pmatrix} \text{condition}$$

Mirror image principles

- PEC conditions satisfied for EM problem + magnetic reflection
- PEC mirror image of a source is its magnetic reflection

$$\begin{pmatrix} \mathbf{J}(\mathbf{r}) \\ \mathbf{M}(\mathbf{r}) \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} -\overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r}) \\ \overline{\overline{C}} \cdot \mathbf{M}(\overline{\overline{C}} \cdot \mathbf{r}) \end{pmatrix}$$

- PMC conditions satisfied for EM problem + electric reflection
- PMC mirror image of a source is its electric reflection

$$\begin{pmatrix} \mathbf{J}(\mathbf{r}) \\ \mathbf{M}(\mathbf{r}) \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \overline{\overline{C}} \cdot \mathbf{J}(\overline{\overline{C}} \cdot \mathbf{r}) \\ -\overline{\overline{C}} \cdot \mathbf{M}(\overline{\overline{C}} \cdot \mathbf{r}) \end{pmatrix}$$

- Image principles transform boundary problems to source problems
- Can be extended to problems with two parallel planes

Medium condition

- PEC/PMC image principle valid if medium invariant in reflection transformation (electric or magnetic)

$$\begin{pmatrix} \overline{\overline{\epsilon}}_c & \overline{\overline{\xi}}_c \\ \overline{\overline{\zeta}}_c & \overline{\overline{\mu}}_c \end{pmatrix} = \overline{\overline{C}} \cdot \begin{pmatrix} \overline{\overline{\epsilon}} & -\overline{\overline{\xi}} \\ -\overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} \cdot \overline{\overline{C}} = \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}$$

- Conditions for medium dyadics

$$\overline{\overline{C}} \cdot \overline{\overline{\epsilon}} \cdot \overline{\overline{C}} = \overline{\overline{\epsilon}}, \quad \overline{\overline{C}} \cdot \overline{\overline{\xi}} \cdot \overline{\overline{C}} = -\overline{\overline{\xi}}, \quad \overline{\overline{C}} \cdot \overline{\overline{\zeta}} \cdot \overline{\overline{C}} = -\overline{\overline{\zeta}}, \quad \overline{\overline{C}} \cdot \overline{\overline{\mu}} \cdot \overline{\overline{C}} = \overline{\overline{\mu}}$$

- Denote transverse dyadics $\overline{\overline{\epsilon}}_t \cdot \mathbf{u} = \mathbf{u} \cdot \overline{\overline{\epsilon}}_t = 0$ and vectors $\mathbf{a}_t \cdot \mathbf{u} = 0$. Medium dyadics must be of the form

$$\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}_t + \epsilon_u \mathbf{uu}, \quad \overline{\overline{\mu}} = \overline{\overline{\mu}}_t + \mu_u \mathbf{uu}, \quad \overline{\overline{\xi}} = \mathbf{a}_t \mathbf{u} + \mathbf{u} \mathbf{b}_t, \quad \overline{\overline{\zeta}} = \mathbf{c}_t \mathbf{u} + \mathbf{u} \mathbf{d}_t$$

- For example, bi-isotropic medium invariant only if $\xi = \zeta = 0$

Object above PEC/PMC plane

- An object in air can be replaced by equivalent polarization source

$$\mathsf{j}_p(\mathbf{r}) = j\omega[\mathsf{M}(\mathbf{r}) - \mathsf{M}_o]\cdot\mathsf{e}(\mathbf{r}), \qquad \mathsf{M}_o = \begin{pmatrix} \epsilon_o & 0 \\ 0 & \mu_o \end{pmatrix}\overline{\overline{I}}$$

- PEC/PMC image of the polarization source obtained through the magnetic/electric reflection transformation

$$\mathsf{j}_{pc}(\mathbf{r}) = \begin{pmatrix} \mp\overline{\overline{C}} & 0 \\ 0 & \pm\overline{\overline{C}} \end{pmatrix}\cdot\mathsf{j}_p(\overline{\overline{C}}\cdot\mathbf{r})$$

$$= j\omega\begin{pmatrix} \mp\overline{\overline{C}} & 0 \\ 0 & \pm\overline{\overline{C}} \end{pmatrix}\cdot[\mathsf{M}(\overline{\overline{C}}\cdot\mathbf{r}) - \mathsf{M}_o]\cdot\mathsf{e}(\overline{\overline{C}}\cdot\mathbf{r}) =$$

$$j\omega\left[\begin{pmatrix} \mp\overline{\overline{C}} & 0 \\ 0 & \pm\overline{\overline{C}} \end{pmatrix}\cdot\mathsf{M}(\overline{\overline{C}}\cdot\mathbf{r})\cdot\begin{pmatrix} \mp\overline{\overline{C}} & 0 \\ 0 & \pm\overline{\overline{C}} \end{pmatrix} - \mathsf{M}_o\right]\cdot\begin{pmatrix} \mp\overline{\overline{C}} & 0 \\ 0 & \pm\overline{\overline{C}} \end{pmatrix}\cdot\mathsf{e}(\overline{\overline{C}}\cdot\mathbf{r})$$

Image of object

- Identifying the image of the obstacle:

$$\mathsf{M}_c(\mathbf{r}) = \begin{pmatrix} \mp\overline{\overline{C}} & 0 \\ 0 & \pm\overline{\overline{C}} \end{pmatrix} \cdot \mathsf{M}(\overline{\overline{C}}\cdot\mathbf{r}) \cdot \begin{pmatrix} \mp\overline{\overline{C}} & 0 \\ 0 & \pm\overline{\overline{C}} \end{pmatrix}$$

$$= \overline{\overline{C}} \cdot \begin{pmatrix} \overline{\overline{\epsilon}}(\overline{\overline{C}}\cdot\mathbf{r}) & -\overline{\overline{\xi}}(\overline{\overline{C}}\cdot\mathbf{r}) \\ -\overline{\overline{\zeta}}(\overline{\overline{C}}\cdot\mathbf{r}) & \overline{\overline{\mu}}(\overline{\overline{C}}\cdot\mathbf{r}) \end{pmatrix} \cdot \overline{\overline{C}}$$

- For example: image of an isotropic half sphere of parameters ϵ, μ on PEC/PMC plane $\mathbf{u}\cdot\mathbf{r} = 0$ is the other half sphere on the plane. Problem of complete sphere and original + image sources

- Image of a chiral half sphere of parameters ϵ, μ, κ is the other half sphere of parameters $\epsilon, \mu, -\kappa$ with changed handedness. The problem does not reduce to that of a complete homogeneous sphere.

Problems

11 Media defined by medium dyadics of the form

$$\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \epsilon & \chi_r\sqrt{\mu\epsilon} \\ \chi_r\sqrt{\mu\epsilon} & \mu \end{pmatrix} \overline{\overline{\alpha}} + \begin{pmatrix} 0 & -j\kappa_r\sqrt{\mu\epsilon} \\ j\kappa_r\sqrt{\mu\epsilon} & 0 \end{pmatrix} \overline{\overline{\beta}}$$

are self dual in some duality transformation. Show that they are also self dual after any affine transformation. Under what condition it is possible to transform $\overline{\overline{\alpha}}$ to the unit dyadic $\overline{\overline{I}}$? How to define the transformation dyadic $\overline{\overline{A}}$?

12 Find the expression for the dipole moment $\mathbf{p}$ of a small dielectric sphere (radius a, permittivity ϵ) at $\mathbf{r} = \mathbf{u}_z h$ in air above a PEC plane at $z = 0$, when the incident field is the plane wave $\mathbf{E}^i(\mathbf{r}) = \mathbf{E}_o e^{-j\mathbf{k}\cdot\mathbf{r}}$. The scatterer can be replaced by the equivalent current dipole $\mathbf{J} = \mathbf{p}\delta(\mathbf{r} - \mathbf{u}_z h)$, $\mathbf{p} = \alpha\mathbf{E}$, $\alpha = j\omega(\epsilon - \epsilon_o)(4\pi a^3/3)$, where $\mathbf{E} = 3\epsilon_o\mathbf{E}'/(\epsilon + 2\epsilon_o)$ is the field inside the scatterer when put in the field $\mathbf{E}'$. h can be assumed large in wavelengths.

S-96.510 Advanced Field Theory
7. Electromagnetic Field Solutions

I.V.Lindell

The Green function

- Green function = field from a point source of unit amplitude

$$L(\nabla)G(\mathbf{r}) = -\delta(\mathbf{r})$$

- Field from distributed source = integral of Green function

$$L(\nabla)f(\mathbf{r}) = g(\mathbf{r}), \qquad f(\mathbf{r}) = -\int G(\mathbf{r}-\mathbf{r}')g(\mathbf{r}')dV'$$

- Green function = mapping source → field
- Scalar source, scalar field, → scalar Green function
- Vector source, vector field, → dyadic Green function
- Six-vector source, six-vector field, → six-dyadic Green function

Green six-dyadic

- Maxwell equations and Maxwell six-dyadic operator $\mathsf{L}(\nabla)$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla\times \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} - j\omega \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ \mathbf{M} \end{pmatrix}$$

$$\mathsf{L}(\nabla)\cdot\mathsf{e}(\mathbf{r}) = \mathsf{j}(\mathbf{r}), \qquad \mathsf{L}(\nabla) = J\nabla\times\overline{\overline{I}} - j\omega\mathsf{M}$$

- Green six-dyadic = four dyadic fields from dyadic unit sources

$$\mathsf{L}(\nabla)\cdot\mathsf{G}(\mathbf{r}) = -\mathsf{I}\delta(\mathbf{r}),$$

$$\mathsf{I} = \begin{pmatrix} \overline{\overline{I}} & 0 \\ 0 & \overline{\overline{I}} \end{pmatrix}, \qquad \mathsf{G}(\mathbf{r}) = \begin{pmatrix} \overline{\overline{G}}_{ee}(\mathbf{r}) & \overline{\overline{G}}_{em}(\mathbf{r}) \\ \overline{\overline{G}}_{me}(\mathbf{r}) & \overline{\overline{G}}_{mm}(\mathbf{r}) \end{pmatrix}$$

- Additional conditions: flow of energy towards infinity (lossless media) or decay of fields towards infinity (lossy media)

- Formal solution $\mathsf{G}(\mathbf{r}) = -\mathsf{L}^{-1}(\nabla)\delta(\mathbf{r})$

Use of Green functions

- Six-vector field from a six-vector source

$$\mathsf{L}(\nabla)\cdot\mathsf{e}(\mathbf{r}) = \mathsf{j}(\mathbf{r}), \qquad \mathsf{L}(\nabla)\cdot\mathsf{G}(\mathbf{r}-\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')\mathsf{I}$$

- can be expressed in terms of the Green six-dyadic

$$\mathsf{e}(\mathbf{r}) = -\int_V \mathsf{G}(\mathbf{r}-\mathbf{r}')\cdot\mathsf{j}(\mathbf{r}')dV'$$

- which is valid outside the sources. Check:

$$\mathsf{L}(\nabla)\cdot\mathsf{e}(\mathbf{r}) = -\int_V [\mathsf{L}(\nabla)\cdot\mathsf{G}(\mathbf{r}-\mathbf{r}')]\cdot\mathsf{j}(\mathbf{r}')dV' = \int_V \delta(\mathbf{r}-\mathbf{r}')\mathsf{j}(\mathbf{r}')dV' = \mathsf{j}(\mathbf{r})$$

- When $\mathbf{r}$ inside the source region V, order of integration and differentiation not interchangeable. Singularity of Green function requires more careful consideration.

Helmholtz operators

- Adjoint Maxwell operators

$$\mathsf{L}^a(\nabla) = \begin{pmatrix} j\omega\overline{\overline{I}} & (\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}})\cdot\overline{\overline{\mu}}^{-1} \\ -(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}})\cdot\overline{\overline{\epsilon}}^{-1} & j\omega\overline{\overline{I}} \end{pmatrix}$$

$$\mathsf{L}_a(\nabla) = \begin{pmatrix} j\omega\overline{\overline{I}} & \overline{\overline{\epsilon}}^{-1}\cdot(\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \\ -\overline{\overline{\mu}}^{-1}\cdot(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) & j\omega\overline{\overline{I}} \end{pmatrix}$$

- diagonalize the Maxwell operator

$$\mathsf{L}^a(\nabla)\cdot\mathsf{L}(\nabla) = \mathsf{L}(\nabla)\cdot\mathsf{L}_a(\nabla) = \begin{pmatrix} \overline{\overline{H}}_e(\nabla) & 0 \\ 0 & \overline{\overline{H}}_m(\nabla) \end{pmatrix},$$

- Helmholtz second-order dyadic operators

$$\overline{\overline{H}}_e(\nabla) = -(\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}})\cdot\overline{\overline{\mu}}^{-1}\cdot(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) + \omega^2\overline{\overline{\epsilon}}$$

$$\overline{\overline{H}}_m(\nabla) = -(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}})\cdot\overline{\overline{\epsilon}}^{-1}\cdot(\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}}) + \omega^2\overline{\overline{\mu}}$$

Two dyadic Green functions

- Define Green six-dyadic in terms of another Green six-dyadic $\mathsf{g}(\mathbf{r})$:

$$\mathsf{G}(\mathbf{r}) = \mathsf{L}_a(\nabla)\mathsf{g}(\mathbf{r})$$

$$\mathsf{L}(\nabla)\cdot\mathsf{G}(\mathbf{r}) = \mathsf{L}(\nabla)\cdot\mathsf{L}_a(\nabla)\cdot\mathsf{g}(\mathbf{r}) = -\mathsf{I}\delta(\mathbf{r})$$

- Diagonal six-dyadic operator $\mathsf{L}(\nabla)\cdot\mathsf{L}_a(\nabla)$ has diagonal solution

$$\mathsf{g}(\mathbf{r}) = \begin{pmatrix} \overline{\overline{G}}_e(\mathbf{r}) & 0 \\ 0 & \overline{\overline{G}}_m(\mathbf{r}) \end{pmatrix}, \qquad \overline{\overline{H}}_{e,m}(\nabla)\cdot\overline{\overline{G}}_{e,m}(\mathbf{r}) = -\overline{\overline{I}}\delta(\mathbf{r})$$

- The original Green six-dyadic is obtained as $\mathsf{G}(\mathbf{r}) = \mathsf{L}_a(\nabla)\mathsf{g}(\mathbf{r})$:

$$\mathsf{G}(\mathbf{r}) = \begin{pmatrix} j\omega\overline{\overline{G}}_e(\mathbf{r}) & \overline{\overline{\epsilon}}^{-1}\cdot(\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}})\cdot\overline{\overline{G}}_m(\mathbf{r}) \\ -\overline{\overline{\mu}}^{-1}\cdot(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}})\cdot\overline{\overline{G}}_e(\mathbf{r}) & j\omega\overline{\overline{G}}_m(\mathbf{r}) \end{pmatrix}$$

- It is sufficient to solve only two Green dyadics $\overline{\overline{G}}_e(\mathbf{r}), \overline{\overline{G}}_m(\mathbf{r})$

Two scalar Green functions

- Green dyadics $\overline{\overline{G}}_e(\mathbf{r})$, $\overline{\overline{G}}_m(\mathbf{r})$ obtained formally as

$$\overline{\overline{G}}_{e,m}(\mathbf{r}) = -\overline{\overline{H}}_{e,m}^{-1}(\nabla)\cdot\delta(\mathbf{r}) = \overline{\overline{H}}_{e,m}^{(2)T}(\nabla)\frac{-1}{\mathrm{det}\overline{\overline{H}}_{e,m}(\nabla)}\delta(\mathbf{r})$$

- Can be solved in terms of two scalar Green functions

$$G_{e,m}(\mathbf{r}) = \frac{-1}{\mathrm{det}\overline{\overline{H}}_{e,m}(\nabla)}\delta(\mathbf{r}),$$

- satisfying scalar fourth-order differential equations

$$\mathrm{det}\overline{\overline{H}}_{e,m}(\nabla)G_{e,m}(\mathbf{r}) = -\delta(\mathbf{r})$$

- Helmholtz Green dyadics can be expressed as

$$\overline{\overline{G}}_{e,m}(\mathbf{r}) = \overline{\overline{H}}_{e,m}^{(2)T}(\nabla)G_{e,m}(\mathbf{r})$$

- In some special cases solving 4th order equations can be avoided

Summary of the general method

- Solve two scalar Green functions satisfying 4th order Helmholtz determinant equations

$$\mathrm{det}\overline{\overline{H}}_{e,m}(\nabla)G_{e,m}(\mathbf{r}) = -\delta(\mathbf{r})$$

- Form two dyadic Green functions which are solutions to 2nd order dyadic Helmholtz equations

$$\overline{\overline{G}}_{e,m}(\mathbf{r}) = \overline{\overline{H}}^{(2)T}_{e,m}(\nabla)G_{e,m}(\mathbf{r}), \qquad \overline{\overline{H}}_{e,m}(\nabla)\cdot\overline{\overline{G}}_{e,m}(\mathbf{r}) = -\overline{\overline{I}}\delta(\mathbf{r})$$

- Form the Green six-dyadic which satisfies the original 1st order six-dyadic equation

$$\mathsf{G}(\mathbf{r}) = \mathsf{L}_a(\nabla)\cdot\begin{pmatrix} \overline{\overline{G}}_e(\mathbf{r}) & 0 \\ 0 & \overline{\overline{G}}_m(\mathbf{r}) \end{pmatrix}, \qquad \mathsf{L}(\nabla)\cdot\mathsf{G}(\mathbf{r}) = -\mathsf{I}\delta(\mathbf{r}),$$

- Difficulty: solutions to fourth-order equations not available
  $\Rightarrow$ works only for some special cases of the bi-anisotropic medium.

Isotropic medium

- Isotropic medium is the simplest special case. Define

$$\overline{\overline{H}}(\nabla) = \mu\overline{\overline{H}}_e(\nabla) = \epsilon\overline{\overline{H}}_m(\nabla) = \overline{\overline{I}}{}^{\times}_{\times}\nabla\nabla + k^2\overline{\overline{I}}$$

$$\mathrm{det}\overline{\overline{H}}(\nabla) = k^2(\nabla^2 + k^2)^2, \qquad \overline{\overline{H}}^{(2)T}(\nabla) = (\nabla^2 + k^2)(\nabla\nabla + k^2\overline{\overline{I}})$$

- Helmholtz Green dyadic solved from 2nd order equation:

$$\overline{\overline{G}}(\mathbf{r}) = -\overline{\overline{H}}^{-1}(\nabla)\delta(\mathbf{r}) = (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla)G(\mathbf{r}) = \mu^{-1}\overline{\overline{G}}_e(\mathbf{r}) = \epsilon^{-1}\overline{\overline{G}}_m(\mathbf{r})$$

$$G(\mathbf{r}) = \frac{-1}{\nabla^2 + k^2}\delta(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r} \qquad \text{(outgoing wave)}$$

- The Green six-dyadic becomes

$$\mathsf{G}(\mathbf{r}) = \begin{pmatrix} \overline{\overline{G}}_{ee}(\mathbf{r}) & \overline{\overline{G}}_{em}(\mathbf{r}) \\ \overline{\overline{G}}_{me}(\mathbf{r}) & \overline{\overline{G}}_{mm}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} j\omega\mu\overline{\overline{G}}(\mathbf{r}) & \nabla G(\mathbf{r}) \times \overline{\overline{I}} \\ -\nabla G(\mathbf{r}) \times \overline{\overline{I}} & j\omega\epsilon\overline{\overline{G}}(\mathbf{r}) \end{pmatrix}$$

Bi-isotropic medium 1

- For bi-isotropic media dyadic Helmholtz operators factorizable

$$\mu\overline{\overline{H}}_e(\nabla) = \epsilon\overline{\overline{H}}_m(\nabla) = \overline{\overline{H}}(\nabla) = -[(\nabla\times\overline{\overline{I}}-j\omega\xi\overline{\overline{I}})\cdot(\nabla\times\overline{\overline{I}}+j\omega\zeta\overline{\overline{I}})-k^2\overline{\overline{I}}]$$

$$\overline{\overline{H}}(\nabla) = -\overline{\overline{L}}_+(\nabla)\cdot\overline{\overline{L}}_-(\nabla) = -\overline{\overline{L}}_-(\nabla)\cdot\overline{\overline{L}}_+(\nabla)$$

$$\overline{\overline{L}}_\pm(\nabla) = \nabla\times\overline{\overline{I}} \mp k\tau_\pm\overline{\overline{I}}, \qquad \tau_\pm = \cos\theta \pm \kappa_r$$

- Only one Green dyadic needed: $\overline{\overline{G}}_e(\mathbf{r}) = \mu\overline{\overline{G}}(\mathbf{r})$, $\overline{\overline{G}}_m(\mathbf{r}) = \epsilon\overline{\overline{G}}(\mathbf{r})$

$$\overline{\overline{G}}(\mathbf{r}) = -\overline{\overline{H}}^{-1}(\nabla)\delta(\mathbf{r}) = \overline{\overline{L}}_+^{-1}(\nabla)\cdot\overline{\overline{L}}_-^{-1}(\nabla)\delta(\mathbf{r})$$

- Dyadic partial fraction expansion applicable in this case

$$\overline{\overline{L}}_+^{-1}(\nabla)\cdot\overline{\overline{L}}_-^{-1}(\nabla) = A_+\overline{\overline{L}}_+^{-1}(\nabla) + A_-\overline{\overline{L}}_-^{-1}(\nabla)$$

$$= [A_+\overline{\overline{L}}_-(\nabla)+A_-\overline{\overline{L}}_+(\nabla)]\cdot\overline{\overline{L}}_+^{-1}(\nabla)\cdot\overline{\overline{L}}_-^{-1}(\nabla) \quad\Rightarrow\quad A_\pm = \pm\frac{1}{2k\cos\theta}$$

Bi-isotropic medium 2

- Green dyadic can be expressed

$$\overline{\overline{G}}(\mathbf{r}) = [A_+\overline{\overline{L}}_+^{-1}(\nabla) + A_-\overline{\overline{L}}_-^{-1}(\nabla)]\delta(\mathbf{r}) = -[A_+\overline{\overline{G}}_+(\mathbf{r}) + A_-\overline{\overline{G}}_-(\mathbf{r})]$$

- in terms of two auxiliary Green dyadics $\overline{\overline{G}}_\pm(\mathbf{r})$

$$\overline{\overline{G}}_\pm = -\overline{\overline{L}}_\pm^{-1}(\nabla)\delta(\mathbf{r}) = -\frac{\overline{\overline{L}}_\pm^{(2)T}}{\det\overline{\overline{L}}_\pm(\nabla)}\delta(\mathbf{r}) = \overline{\overline{L}}_\pm^{(2)T}(\nabla)G_\pm(\mathbf{r}),$$

$$\overline{\overline{L}}_\pm^{(2)T}(\nabla) = \nabla\nabla \pm k\tau_\pm\nabla\times\overline{\overline{I}} + k^2\tau_\pm^2\overline{\overline{I}}$$

- Scalar Green functions $G_\pm(\mathbf{r})$ satisfy second-order Helmholtz equations

$$\det\overline{\overline{L}}_\pm(\nabla)G_\pm(\mathbf{r}) = \mp k\tau_\pm(\nabla^2 + k^2\tau_\pm^2)G_\pm(\mathbf{r}) = -\delta(\mathbf{r})$$

Bi-isotropic medium 3

- Solution to the scalar Helmholtz equation (outgoing waves assume $k\tau_\pm > 0$)

$$(\nabla^2 + k^2\tau_\pm^2)G_\pm(\mathbf{r}) = \pm\frac{1}{k\tau_\pm}\delta(\mathbf{r}), \quad G_\pm(\mathbf{r}) = \mp\frac{e^{-jk\tau_\pm r}}{4\pi k\tau_\pm r}$$

- Solutions to Green dyadics

$$\overline{\overline{G}}_e(\mathbf{r}) = \mu\overline{\overline{G}}(\mathbf{r}), \quad \overline{\overline{G}}_m(\mathbf{r}) = \epsilon\overline{\overline{G}}(\mathbf{r}),$$

$$\overline{\overline{G}}(\mathbf{r}) = \frac{1}{2k\cos\theta}\left(\overline{\overline{L}}_+^{(2)T}(\nabla)\frac{e^{-jk\tau_+ r}}{4\pi k\tau_+ r} + \overline{\overline{L}}_-^{(2)T}(\nabla)\frac{e^{-jk\tau_- r}}{4\pi k\tau_- r}\right)$$

- Two terms correspond to two self-dual fields
- This result can also be also obtained through the general method with some more effort

Self-dual medium 1

- Self-dual medium is a generalization of the bi-isotropic medium

$$\mathsf{M} = \begin{pmatrix} \epsilon & \sin\theta\sqrt{\mu\epsilon} \\ \sin\theta\sqrt{\mu\epsilon} & \mu \end{pmatrix} \overline{\overline{\alpha}} + \begin{pmatrix} 0 & -j\sqrt{\mu\epsilon} \\ j\sqrt{\mu\epsilon} & 0 \end{pmatrix} \overline{\overline{\kappa}}_r$$

- Denote $\mu\overline{\overline{H}}_e(\nabla) = \epsilon\overline{\overline{H}}_m(\nabla) = \overline{\overline{H}}(\nabla)$ which can be factorized as

$$\overline{\overline{H}}(\nabla) = -(\nabla\times\overline{\overline{I}} - j\omega\overline{\overline{\xi}})\cdot\overline{\overline{\alpha}}^{-1}\cdot(\nabla\times\overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) + k^2\overline{\overline{\alpha}}$$

$$= -\overline{\overline{L}}_+(\nabla)\cdot\overline{\overline{\alpha}}^{-1}\cdot\overline{\overline{L}}_-(\nabla) = -\overline{\overline{L}}_-(\nabla)\cdot\overline{\overline{\alpha}}^{-1}\cdot\overline{\overline{L}}_+(\nabla)$$

$$\overline{\overline{L}}_\pm(\nabla) = \nabla\times\overline{\overline{I}} \mp k\overline{\overline{\tau}}_\pm, \quad \overline{\overline{\tau}}_\pm = \cos\theta\overline{\overline{\alpha}} \pm \overline{\overline{\kappa}}_r$$

- Formal solution for $\overline{\overline{G}}_e(\mathbf{r}) = \mu\overline{\overline{G}}(\mathbf{r})$, $\overline{\overline{G}}_m(\mathbf{r}) = \epsilon\overline{\overline{G}}(\mathbf{r})$ from

$$\overline{\overline{G}}(\mathbf{r}) = -\overline{\overline{H}}^{-1}(\nabla)\delta(\mathbf{r}) = \overline{\overline{L}}_+^{-1}(\nabla)\cdot\overline{\overline{\alpha}}\cdot\overline{\overline{L}}_-^{-1}(\nabla)\delta(\mathbf{r})$$

Self-dual medium 2

- Dyadic partial fraction expansion possible also in this case

$$\overline{\overline{L}}_+^{-1}(\nabla)\cdot\overline{\overline{\alpha}}\cdot\overline{\overline{L}}_-^{-1}(\nabla) = A_+\overline{\overline{L}}_+^{-1}(\nabla) + A_-\overline{\overline{L}}_-^{-1}(\nabla)$$

$$= \overline{\overline{L}}_+^{-1}(\nabla)\cdot[A_+\overline{\overline{L}}_-(\nabla) + A_-\overline{\overline{L}}_+(\nabla)]\cdot\overline{\overline{L}}_-^{-1}(\nabla)$$

$$A_+\overline{\overline{L}}_-(\nabla) + A_-\overline{\overline{L}}_+(\nabla) = \overline{\overline{\alpha}} \quad\Rightarrow\quad A_\pm = \pm\frac{1}{2k\cos\theta}$$

- Solution in terms of auxiliary Green functions

$$\overline{\overline{G}}(\mathbf{r}) = \overline{\overline{L}}_+^{-1}(\nabla)\cdot\overline{\overline{\alpha}}\cdot\overline{\overline{L}}_-^{-1}(\nabla)\delta(\mathbf{r}) = -\frac{1}{2k\cos\theta}[\overline{\overline{G}}_+(\mathbf{r}) - \overline{\overline{G}}_-(\mathbf{r})]$$

$$\overline{\overline{G}}_\pm(\mathbf{r}) = -\overline{\overline{L}}_\pm^{-1}(\nabla)\delta(\mathbf{r}) = -\frac{\overline{\overline{L}}_\pm^{(2)T}(\nabla)}{\mathrm{det}\overline{\overline{L}}_\pm(\nabla)}\delta(\mathbf{r}) = \overline{\overline{L}}_\pm^{(2)T}(\nabla)G_\pm(\mathbf{r})$$

Self-dual medium 3

- Auxiliary Green dyadics solved from

$$\mathrm{det}\overline{\overline{L}}_\pm(\nabla)G_\pm(\mathbf{r}) = \mathrm{det}(\nabla\times\overline{\overline{I}} \mp k\overline{\overline{\tau}}_\pm)G_\pm(\mathbf{r}) = -\delta(\mathbf{r})$$

- Expand $\overline{\overline{\tau}}_\pm$ in symmetric and antisymmetric parts:

$$\overline{\overline{\tau}}_\pm = \overline{\overline{S}}_\pm + \mathbf{a}_\pm \times \overline{\overline{I}}$$

$$\mathrm{det}\overline{\overline{L}}_\pm(\nabla) = \mp k[\overline{\overline{S}}_\pm : (\nabla \mp k\mathbf{a}_\pm)(\nabla \mp k\mathbf{a}_\pm) + k^2\mathrm{det}\overline{\overline{S}}_\pm]$$

- Helmholtz equation becomes

$$[\overline{\overline{S}}_\pm : (\nabla \mp k\mathbf{a}_\pm)(\nabla \mp k\mathbf{a}_\pm) + k^2\mathrm{det}\overline{\overline{S}}_\pm][\mp kG_\pm(\mathbf{r})] = -\delta(\mathbf{r})$$

$$[\overline{\overline{S}}_\pm : \nabla\nabla + k^2\mathrm{det}\overline{\overline{S}}_\pm][\mp ke^{\mp k\mathbf{a}_\pm\cdot\mathbf{r}}G_\pm(\mathbf{r})] = -\delta(\mathbf{r})$$

- This can be transformed to a more familiar form

Self-dual medium 4

- Apply affine transformation

$$\nabla' = \overline{\overline{S}}_\pm^{1/2} \cdot \nabla, \qquad \mathbf{r}' = \overline{\overline{S}}_\pm^{-1/2} \cdot \mathbf{r}, \qquad \delta(\mathbf{r}') = \delta(\mathbf{r}) \sqrt{\mathrm{det}\overline{\overline{S}}_\pm}$$

$$(\nabla'^2 + k^2 \mathrm{det}\overline{\overline{S}}_\pm) \left[\mp k e^{\mp k \mathbf{a}_\pm \cdot \mathbf{r}} G_\pm(\mathbf{r}) \sqrt{\mathrm{det}\overline{\overline{S}}_\pm} \right] = -\delta(\mathbf{r}')$$

- Solution with $r' = \sqrt{\mathbf{r}' \cdot \mathbf{r}'} = \sqrt{\overline{\overline{S}}_\pm^{-1} : \mathbf{rr}}$

$$k e^{\mp k \mathbf{a}_\pm \cdot \mathbf{r}} G_\pm(\mathbf{r}) \sqrt{\mathrm{det}\overline{\overline{S}}_\pm} = \frac{e^{-jk\sqrt{\mathrm{det}\overline{\overline{S}}_\pm} r'}}{4\pi r'} = \frac{e^{-jk\sqrt{\overline{\overline{S}}_\pm^{(2)} : \mathbf{rr}}}}{4\pi \sqrt{\overline{\overline{S}}_\pm^{-1} : \mathbf{rr}}}$$

- Solutions for the two auxiliary scalar Green functions

$$G_\pm(\mathbf{r}) = \mp e^{\pm k \mathbf{a}_\pm \cdot \mathbf{r}} \frac{e^{-jkD_\pm}}{4\pi k D_\pm}, \qquad D_\pm(\mathbf{r}) = \sqrt{\overline{\overline{S}}_\pm^{(2)} : \mathbf{rr}}$$

Self-dual medium 5

- The dyadic Green functions $\overline{\overline{G}}_e(\mathbf{r}) = \mu\overline{\overline{G}}(\mathbf{r})$, $\overline{\overline{G}}_m(\mathbf{r}) = \epsilon\overline{\overline{G}}(\mathbf{r})$ are finally obtained from

$$\overline{\overline{G}}(\mathbf{r}) = \frac{-1}{2k\cos\theta}[\overline{\overline{G}}_+(\mathbf{r}) - \overline{\overline{G}}_-(\mathbf{r})]$$

$$= \frac{1}{2k\cos\theta}[-\overline{\overline{L}}_+^{(2)T}(\nabla)G_+(\mathbf{r}) + \overline{\overline{L}}_-^{(2)T}(\nabla)G_-(\mathbf{r})],$$

- by inserting

$$\mp\overline{\overline{L}}_\pm^{(2)T}(\nabla)G_\pm(\mathbf{r}) = [-(\nabla \mp k\mathbf{a}_\pm) \times \overline{\overline{I}} \mp \overline{\overline{S}}_\pm]^{(2)} e^{\pm k\mathbf{a}_\pm\cdot\mathbf{r}} \frac{e^{-jkD_\pm}}{4\pi kD_\pm}$$

$$= e^{\pm k\mathbf{a}_\pm\cdot\mathbf{r}}[\nabla\nabla \pm k(\overline{\overline{S}}_\pm \cdot \nabla) \times \overline{\overline{I}} + k^2\overline{\overline{S}}_\pm^{(2)}]\frac{e^{-jkD_\pm}}{4\pi kD_\pm}$$

- Again the two terms correspond to the self-dual fields
- $\mathbf{a}_\pm = 0$, $\overline{\overline{S}}_\pm = \tau_\pm\overline{\overline{I}}$ gives the previous result for bi-isotropic space

Uniaxial medium 1

- Helmholtz dyadic factorizable only in self-dual media. However, Helmholtz determinant factorizable in uniaxial anisotropic medium

$$\overline{\overline{\epsilon}} = \epsilon_z \mathbf{u}_z\mathbf{u}_z + \epsilon_t \overline{\overline{I}}_t, \qquad \overline{\overline{\mu}} = \mu_z \mathbf{u}_z\mathbf{u}_z + \mu_t \overline{\overline{I}}_t, \qquad \overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$$

- Electric Green dyadic

$$\overline{\overline{G}}_e(\mathbf{r}) = -\overline{\overline{H}}_e^{-1}(\nabla)\delta(\mathbf{r}), \quad \overline{\overline{H}}_e(\nabla) = \overline{\overline{\mu}}^{-1} {}^\times_\times \nabla\nabla + \omega^2\overline{\overline{\epsilon}}$$

$$\overline{\overline{G}}_e(\mathbf{r}) = \overline{\overline{H}}_e^{(2)T}(\nabla)G(\mathbf{r}), \qquad G(\mathbf{r}) = -\frac{1}{\mathrm{det}\overline{\overline{H}}_e(\nabla)}\delta(\mathbf{r})$$

- The operator $\mathrm{det}\overline{\overline{H}}_e(\nabla)$ can be factorized (after some effort):

$$\mathrm{det}\overline{\overline{H}}_e(\nabla) = \frac{k_t^2}{\mathrm{det}\overline{\overline{\mu}}} H_\mu(\nabla)H_\epsilon(\nabla), \qquad k_t^2 = \omega^2\mu_t\epsilon_t$$

$$H_\epsilon(\nabla) = \nabla_t^2 + \frac{\epsilon_z}{\epsilon_t}(\partial_z^2 + k_t^2), \qquad H_\mu(\nabla) = \nabla_t^2 + \frac{\mu_z}{\mu_t}(\partial_z^2 + k_t^2)$$

Uniaxial medium 2

- Scalar Green function

$$G(\mathbf{r}) = -\frac{\det\overline{\overline{\mu}}}{k_t^2}\frac{1}{H_\mu(\nabla)H_\epsilon(\nabla)}\delta(\mathbf{r}) = \frac{\det\overline{\overline{\mu}}}{\omega^2(\epsilon_z\mu_t - \epsilon_t\mu_z)}[g_\mu(\mathbf{r}) - g_\epsilon(\mathbf{r})],$$

$$g_\alpha(\mathbf{r}) = -\frac{1}{(\partial_z^2 + k_t^2)(\nabla_t^2 + \frac{\alpha_z}{\alpha_t}(\partial_z^2 + k_t^2))}\delta(\mathbf{r}), \qquad \alpha = \mu, \epsilon$$

- It can be shown that $g_\alpha(\mathbf{r})$ can be solved analytically as

$$g_\alpha(\mathbf{r}) = \frac{j}{8\pi k_t \alpha_t}[E_1(jk_t(D_\alpha - z))e^{-jk_t z} + E_1(jk_t(D_\alpha + z))e^{jk_t z}],$$

- $E_1(x)$ = exponential integral function, $D_\alpha(\mathbf{r})$ = distance function

$$E_1(x) = \int_x^\infty \frac{e^{-y}}{y}dy, \qquad D_\alpha = \sqrt{\frac{\alpha_z}{\alpha_t}\rho^2 + z^2}$$

Uniaxial medium 3

- The electric Green dyadic becomes

$$\overline{\overline{G}}_e(\mathbf{r}) = \overline{\overline{H}}_e^{(2)T}(\nabla)G(\mathbf{r}) = \frac{\mathrm{det}\overline{\overline{\mu}}}{\omega^2(\epsilon_z\mu_t - \epsilon_t\mu_z)}\overline{\overline{H}}_e^{(2)T}(\nabla)[g_\mu(\mathbf{r}) - g_\epsilon(\mathbf{r})]$$

- After differentiations (quite tedious)

$$\overline{\overline{G}}_e(\mathbf{r}) = \left(\epsilon_z\overline{\overline{\epsilon}}^{-1} + \frac{1}{k_t^2}\nabla\nabla\right)\frac{e^{-jk_tD_\epsilon}}{4\pi D_\epsilon} + \nabla\times\frac{j\tau\mathbf{u}_z\mathbf{q}}{4\pi k_t}$$

$$\tau = e^{-jk_tD_\epsilon} - e^{-jk_tD_\mu}, \qquad \mathbf{q} = \frac{\mathbf{u}_z\times\mathbf{r}}{(\mathbf{u}_z\times\mathbf{r})^2}$$

- Solution not singular at z axis $\mathbf{u}_z\times\mathbf{r} = 0$ because $\tau\mathbf{q}$ is finite
- Solution generalizable to $\overline{\overline{\xi}} = \xi_z\mathbf{u}_z\mathbf{u}_z$, $\overline{\overline{\zeta}} = \zeta_z\mathbf{u}_z\mathbf{u}_z$.
- Factorizability of $\mathrm{det}\overline{\overline{H}}(\nabla)$ does not warrant analytic solution for Green dyadic!

A class of anisotropic media 1

- Helmholtz determinant operator factorizable for the class of anisotropic media defined by $\overline{\overline{\mu}} = \tau\overline{\overline{\epsilon}}^T$, $\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$

$$\det\overline{\overline{H}}_e(\nabla) = \det(\overline{\overline{\mu}}^{-1}{}^\times_\times\nabla\nabla + \omega^2\overline{\overline{\epsilon}}) = \det(\frac{1}{\tau}\overline{\overline{\epsilon}}^{-1T}{}^\times_\times\nabla\nabla + \omega^2\overline{\overline{\epsilon}})$$

$$= \frac{\omega^2}{\tau^2\det\overline{\overline{\epsilon}}}(\overline{\overline{\epsilon}}:\nabla\nabla)^2 + \frac{\omega^4}{\tau}\overline{\overline{\epsilon}}:\nabla\nabla + \omega^6\det\overline{\overline{\epsilon}} = \frac{\omega^2}{\tau^2\det\overline{\overline{\epsilon}}}(\overline{\overline{\epsilon}}:\nabla\nabla + \omega^2\tau\det\overline{\overline{\epsilon}})^2$$

- Leads to a second-order PDE because also $\overline{\overline{H}}_e^{(2)}(\nabla)$ factorizable:

$$\overline{\overline{H}}_e^{(2)}(\nabla) = \frac{1}{\tau^2\det\overline{\overline{\epsilon}}}(\overline{\overline{\epsilon}}:\nabla\nabla)\nabla\nabla + \frac{\omega^2}{\tau\det\overline{\overline{\epsilon}}}(\overline{\overline{\epsilon}}^{(2)}{}^\times_\times\nabla\nabla){}^\times_\times\overline{\overline{\epsilon}} + \omega^4\overline{\overline{\epsilon}}^{(2)}$$

$$= \frac{1}{\tau^2\det\overline{\overline{\epsilon}}}(\overline{\overline{\epsilon}}:\nabla\nabla + \omega^2\tau\det\overline{\overline{\epsilon}})(\nabla\nabla + \omega^2\tau\overline{\overline{\epsilon}}^{(2)})$$

$$\Rightarrow \quad \overline{\overline{H}}_e^{-1}(\nabla) = \frac{1}{\omega^2}\frac{\nabla\nabla + \omega^2\tau\overline{\overline{\epsilon}}^{(2)T}}{\overline{\overline{\epsilon}}:\nabla\nabla + \omega^2\tau\det\overline{\overline{\epsilon}}}$$

A class of anisotropic media 2

- Solution of the scalar Green function ($\overline{\overline{\epsilon}}_s$ = symmetric part of $\overline{\overline{\epsilon}}$)

$$G(\mathbf{r}) = -\frac{1}{\overline{\overline{\epsilon}}_s : \nabla\nabla + k^2}\delta(\mathbf{r}) = \frac{e^{-jkD}}{4\pi\sqrt{\mathrm{det}\overline{\overline{\epsilon}}_s}D}, \qquad D = \sqrt{\overline{\overline{\epsilon}}_s^{-1} : \mathbf{rr}}$$

- Electric Green dyadic obtained in analytic form

$$\overline{\overline{G}}_e(\mathbf{r}) = -\overline{\overline{H}}_e^{-1}(\nabla)\delta(\mathbf{r}) = -\frac{1}{\omega^2}\frac{\nabla\nabla + \omega^2\tau\overline{\overline{\epsilon}}^{(2)T}}{\overline{\overline{\epsilon}} : \nabla\nabla + \omega^2\tau\mathrm{det}\overline{\overline{\epsilon}}}\delta(\mathbf{r})$$

$$= (\nabla\nabla + \omega^2\tau\overline{\overline{\epsilon}}^{(2)T})\frac{e^{-j\omega\sqrt{\tau\mathrm{det}\overline{\overline{\epsilon}}}\sqrt{\overline{\overline{\epsilon}}_s^{-1}:\mathbf{rr}}}}{4\pi\omega^2\sqrt{\mathrm{det}\overline{\overline{\epsilon}}_s}\sqrt{\overline{\overline{\epsilon}}_s^{-1} : \mathbf{rr}}}$$

- Check: isotropic case $\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}_s = \epsilon\overline{\overline{I}}$, $\tau = \mu/\epsilon$:

  $\Rightarrow\ \overline{\overline{G}}_e = \mu(\overline{\overline{I}} + \nabla\nabla/k^2)(e^{-jkr}/4\pi r), \quad k = \omega\sqrt{\mu\epsilon}$

Problems

13 Derive the electric Green dyadic for a bi-anisotropic medium whose medium dyadics are of the form

$$\overline{\overline{\epsilon}} = \epsilon\overline{\overline{S}}, \quad \overline{\overline{\mu}} = \mu\overline{\overline{S}}, \quad \overline{\overline{\xi}} = \mathbf{x}\times\overline{\overline{I}}, \quad \overline{\overline{\zeta}} = \mathbf{z}\times\overline{\overline{I}},$$

where $\overline{\overline{S}}$ is a complete symmetric dyadic, $\mathbf{x}, \mathbf{z}$ two vectors and ϵ, μ two scalars.

14 Show that the Helmholtz determinant operators for the anisotropic medium defined by the medium dyadics of the form

$$\overline{\overline{\epsilon}} = \epsilon\overline{\overline{A}}, \quad \overline{\overline{\mu}} = \mu\overline{\overline{A}}^T + \alpha\mathbf{ab}, \quad \overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$$

can be factorized. $\overline{\overline{A}}$ is a dyadic, $\mathbf{a}, \mathbf{b}$ two vectors and ϵ, μ, α three scalars.

S-96.510 Advanced Field Theory
8. Green dyadic singularities and complex-space sources

I.V.Lindell

Fields outside sources

- Isotropic medium Green six-dyadic

$$\begin{pmatrix} \overline{\overline{G}}_{ee}(\mathbf{r}) & \overline{\overline{G}}_{em}(\mathbf{r}) \\ \overline{\overline{G}}_{me}(\mathbf{r}) & \overline{\overline{G}}_{mm}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} j\omega\mu\overline{\overline{G}}(\mathbf{r}) & \nabla G(\mathbf{r}) \times \overline{\overline{I}} \\ -\nabla G(\mathbf{r}) \times \overline{\overline{I}} & j\omega\epsilon\overline{\overline{G}}(\mathbf{r}) \end{pmatrix}$$

$$G(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r}, \qquad \overline{\overline{G}}(\mathbf{r}) = (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla)G(\mathbf{r})$$

Fields outside source area V

$$\begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix} = -\int_V \begin{pmatrix} \overline{\overline{G}}_{ee}(\mathbf{r}-\mathbf{r}') & \overline{\overline{G}}_{em}(\mathbf{r}-\mathbf{r}') \\ \overline{\overline{G}}_{me}(\mathbf{r}-\mathbf{r}') & \overline{\overline{G}}_{mm}(\mathbf{r}-\mathbf{r}') \end{pmatrix} \cdot \begin{pmatrix} \mathbf{J}(\mathbf{r}') \\ \mathbf{M}(\mathbf{r}') \end{pmatrix} dV'$$

- For example, electric field from electric source

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int_V \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}') dV'$$

Singularity of Green dyadic

- When $D = |\mathbf{r} - \mathbf{r}'|$ small, $G(D) \approx 1/4\pi D$ large but integrable:

$$\int \frac{1}{4\pi r} dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{1}{4\pi r} r^2 dr d\theta d\varphi = \int_0^a r dr = \frac{a^2}{2}$$

- Green dyadic more singular $|\overline{\overline{G}}(D)| \approx |\nabla\nabla G(\nabla)| \approx |\mathbf{u}_r\mathbf{u}_r/D^3|$
  Not integrable! How to compute fields inside the source $\mathbf{r} \in V$?

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int_V (\overline{\overline{I}} + \frac{1}{k^2}\nabla\nabla)G(\mathbf{r} - \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')dV'$$

$$\mathbf{H}(\mathbf{r}) = -\int_V \nabla G(\mathbf{r} - \mathbf{r}') \times \mathbf{J}(\mathbf{r}')dV'$$

- Solution: split $V = V_\delta + V_1$. Small volume V_δ around field point $\mathbf{r}$ containing the singularity is computed separately.

Principal-value integrals

- Outside $\mathbf{r}=0$ we have $(\nabla^2+k^2)G(\mathbf{r})=0$ and

$$[\overline{\overline{I}}+\frac{1}{k^2}\nabla\nabla]G(\mathbf{r})=\frac{1}{k^2}(-\nabla^2\overline{\overline{I}}+\nabla\nabla)G(\mathbf{r})=-\frac{1}{k^2}\overline{\overline{I}}{}^{\times}_{\times}\nabla\nabla G(\mathbf{r})$$

- The electric field integral over $V_1=V-V_\delta$ is well defined:

$$\mathbf{E}_1(\mathbf{r})=-\frac{1}{j\omega\epsilon}\overline{\overline{I}}{}^{\times}_{\times}\int\limits_{V_1}\nabla\nabla G(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')dV'$$

- The limit $V_\delta\to 0$ is denoted as principal-value integral $\mathcal{PV}_\delta$

$$\mathbf{E}_1(\mathbf{r})\to-\frac{1}{j\omega\epsilon}\overline{\overline{I}}{}^{\times}_{\times}\mathcal{PV}_\delta\int\limits_{V}[\nabla'\nabla'G(\mathbf{r}-\mathbf{r}')]\cdot\mathbf{J}(\mathbf{r}')dV'$$

- Total field $\mathbf{E}=\mathbf{E}_1+\mathbf{E}_\delta$. The contribution $\mathbf{E}_\delta$ from $V_\delta\to 0$ is obtained separately by assuming that the current is constant $\mathbf{J}(\mathbf{r}')=\mathbf{J}(\mathbf{r})$ over the small volume V_δ.

Field from constant current

- If constant current $\mathbf{J}_c$ fills all space V_∞, fields are constant $\mathbf{E}_c, \mathbf{H}_c$

$$\nabla \times \mathbf{H}_c = 0 = j\omega\epsilon \mathbf{E}_c + \mathbf{J}_c, \quad \Rightarrow \quad \mathbf{E}_c = -\frac{\mathbf{J}_c}{j\omega\epsilon}, \quad \mathbf{H}_c = 0$$

- No singularity for field inside the source!

- Field from volume V_δ of constant current $\mathbf{J}_c$ around point $\mathbf{r}$: $\mathbf{E}_\delta(\mathbf{r}) = \mathbf{E}_c - \mathbf{E}_1(\mathbf{r})$. Limit $V_\delta \to 0 \;\Rightarrow$ principal-value integral:

$$\mathbf{E}_\delta(\mathbf{r}) = \mathbf{E}_c + \frac{1}{j\omega\epsilon} \overline{\overline{I}}{}^{\times}_{\times} \mathcal{PV}_\delta \int_{V_\infty} \nabla'\nabla' G(\mathbf{r} - \mathbf{r}')dV' \cdot \mathbf{J}_c$$

$$= -\frac{1}{j\omega\epsilon}(\overline{\overline{I}} - \overline{\overline{I}}{}^{\times}_{\times}\overline{\overline{L}}_\delta) \cdot \mathbf{J}_c, \qquad \overline{\overline{L}}_\delta = \mathcal{PV}_\delta \int_{V_\infty} \nabla'\nabla' G(\mathbf{r} - \mathbf{r}')dV'$$

- $\overline{\overline{L}}_\delta$ = depolarization dyadic depends on the volume V_δ

Depolarization dyadic

- Depolarization dyadic $\overline{\overline{L}}_\delta$ as surface integral from Gaussian law

$$\overline{\overline{L}}_\delta = \mathcal{PV}_\delta \int_{V_\infty} \nabla'\nabla' G(\mathbf{r}-\mathbf{r}')dV' = -\lim \oint_{S_\delta} \mathbf{n}'\nabla' G(\mathbf{r}-\mathbf{r}')dS'$$

- S_δ = surface of the small volume V_δ, $\mathbf{n}'$ = outside unit normal
Limit $\delta \to 0$ static approximation valid $G(\mathbf{R}) \to 1/4\pi R$
$\mathbf{R}' = \mathbf{r}' - \mathbf{r}$, $dS' = R'^2 d\Omega'/\mathbf{n}'\cdot\mathbf{u}'_R$

$$\overline{\overline{L}}_\delta = -\oint_{S_\delta} \mathbf{n}'\nabla' \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}dS' = \frac{1}{4\pi}\oint_{4\pi} \frac{\mathbf{n}'\mathbf{u}'_R}{\mathbf{n}'\cdot\mathbf{u}'_R}d\Omega'$$

- Depolarization dyadic $\overline{\overline{L}}_\delta$ is symmetric and satisfies $\overline{\overline{L}}_\delta : \overline{\overline{I}} = 1$
Independent of size of S_δ, depends on form of S_δ and location of origin of $\mathbf{R}'$.

Singularity of Green dyadic

- Field inside a small volume $V_\delta \to 0$ of constant current $\mathbf{J}_c$

$$\mathbf{E}_\delta(\mathbf{r}) = -\frac{1}{j\omega\epsilon}(\overline{\overline{I}} - \overline{\overline{L}}_\delta {}^{\times}_{\times} \overline{\overline{I}}) \cdot \mathbf{J}_c = -\frac{1}{j\omega\epsilon}\overline{\overline{L}}_\delta \cdot \mathbf{J}_c$$

- Applied to field integral for point $\mathbf{r}$ inside the source region:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_\delta(\mathbf{r}) + \mathbf{E}_1(\mathbf{r}) = -\frac{1}{j\omega\epsilon}\overline{\overline{L}}_\delta \cdot \mathbf{J}(\mathbf{r}) - j\omega\mu\mathcal{PV}_\delta \int_V \overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')dV'$$

$$= -j\omega\mu \int_V [\mathcal{PV}_\delta \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') - \frac{1}{k^2}\overline{\overline{L}}_\delta \delta(\mathbf{r}-\mathbf{r}')] \cdot \mathbf{J}(\mathbf{r}')dV'$$

- The Green dyadic consists of a regular part (principal value) and a singular delta-function part

$$\overline{\overline{G}}(\mathbf{r}-\mathbf{r}') = \mathcal{PV}_\delta \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') - \frac{1}{k^2}\overline{\overline{L}}_\delta \delta(\mathbf{r}-\mathbf{r}')$$

Different contracting surfaces

- Spherical S_δ, origin at center: $\mathbf{n}' = \mathbf{u}'_R$

$$\overline{\overline{L}}_\delta = \frac{1}{4\pi}\oint_{4\pi} \frac{\mathbf{n}'\mathbf{u}'_R}{\mathbf{n}'\cdot\mathbf{u}'_R} d\Omega' = \frac{1}{4\pi}\oint_{4\pi} \mathbf{u}'_R\mathbf{u}'_R d\Omega' = \frac{1}{3}\overline{\overline{I}}$$

- Symmetric surface $\Rightarrow \overline{\overline{L}}_\delta = \alpha\overline{\overline{I}}$. From $\overline{\overline{L}}_\delta : \overline{\overline{I}} = 1 \;\; \Rightarrow \;\; \alpha = 1/3$. Valid for cube, tetrahedron, etc when field point at center

- Long circular cylinder, axis $\mathbf{u}_z$

$$\overline{\overline{L}}_\delta = \frac{1}{2}(\overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z)$$

- Ellipsoid, L_x, L_y, L_z depolarizing factors, $A = 1/(L_x + L_y + L_z)$

$$\overline{\overline{L}}_\delta = A(L_x\mathbf{u}_x\mathbf{u}_x + L_y\mathbf{u}_y\mathbf{u}_y + L_z\mathbf{u}_z\mathbf{u}_z)$$

Example of depolarization dyadic

- Spheroid with normal vector $\mathbf{n}$

$$f(\mathbf{r}) = \frac{\rho^2}{b^2} + \frac{z^2}{a^2} = 1, \qquad \mathbf{n} = \frac{\nabla f(\mathbf{r})}{|\nabla f(\mathbf{r})|} = \frac{a^2\boldsymbol{\rho} + b^2 z\mathbf{u}_z}{\sqrt{a^4\rho^2 + b^4 z^2}}$$

- Has one special direction $\mathbf{u}_z$
  $\Rightarrow$ dyadic must be of uniaxial form $\overline{\overline{L}}_\delta = L_z\mathbf{u}_z\mathbf{u}_z + L_t\overline{\overline{I}}_t$

- $\overline{\overline{L}}_\delta : \overline{\overline{I}} = 1 \;\Rightarrow\; L_z = 1 - 2L_t$, only one parameter to be found

$$\overline{\overline{L}}_\delta = \frac{1}{4\pi}\oint_{4\pi} \frac{\mathbf{n}'\mathbf{u}_r'}{\mathbf{n}'\cdot\mathbf{u}_r'} d\Omega' = L_z\mathbf{u}_z\mathbf{u}_z + L_t\overline{\overline{I}}_t, \quad \Rightarrow \quad \overline{\overline{L}}_\delta : \mathbf{u}_z\mathbf{u}_z = L_z$$

$$L_z = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{(\nabla f\cdot\mathbf{u}_z)(\mathbf{u}_z\cdot\mathbf{r})}{\nabla f\cdot\mathbf{r}} \sin\theta d\theta d\varphi = \int_0^{\pi/2} \frac{b^2\cos^2\theta\sin\theta d\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

- Can be solved in terms of elementary functions.

Scattering problem

- Dielectric scatterer V_s set in incident field $\mathbf{E}^i(\mathbf{r})$
- Scatterer can be replaced by equivalent polarization current $\mathbf{J}_p(\mathbf{r})$:

$$\nabla\times\mathbf{H}(\mathbf{r}) = j\omega\epsilon(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}) = j\omega\epsilon_o\cdot\mathbf{E}(\mathbf{r}) + \mathbf{J}_p(\mathbf{r})$$

$$\Rightarrow \quad \mathbf{J}_p(\mathbf{r}) = j\omega\epsilon_o[\epsilon_r(\mathbf{r}) - 1]\cdot\mathbf{E}(\mathbf{r})$$

- Scattered field = field from polarization current

$$\mathbf{E}^{sc}(\mathbf{r}) = -j\omega\mu_o\int_{V_s}\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}_p(\mathbf{r}')dV' = k_o^2\int_{V_s}[\epsilon_r(\mathbf{r}')-1]\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}')dV'$$

- Total field $\mathbf{E}(\mathbf{r}) = \mathbf{E}^i(\mathbf{r}) + \mathbf{E}^{sc}(\mathbf{r})$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^i(\mathbf{r}) + k_o^2\int_{V_s}[\epsilon_r(\mathbf{r}') - 1]\overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}')dV'$$

- $\mathbf{r}\in V_s \;\Rightarrow$ integral equation for the total field inside the scatterer.

Singularity extraction

- Insert $\overline{\overline{G}} = \mathcal{PV}_\delta \overline{\overline{G}}(\mathbf{r}) - \overline{\overline{L}}_\delta \delta(\mathbf{r})/k_o^2 \;\Rightarrow$ no singularity in integral:

$$[\overline{\overline{I}} + (\epsilon_r(\mathbf{r}) - 1)\overline{\overline{L}}_\delta] \cdot \mathbf{E}(\mathbf{r}) = \mathbf{E}^i(\mathbf{r}) + k_o^2 \mathcal{PV}_\delta \int_{V_s} [\epsilon_r(\mathbf{r}') - 1] \overline{\overline{G}}(\mathbf{r} - \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dV'$$

- For example, constant ϵ_r and choose symmetric S_δ with $\overline{\overline{L}}_\delta = \overline{\overline{I}}/3$
Simplified integral equation

$$\frac{\epsilon_r + 2}{3} \mathbf{E}(\mathbf{r}) = \mathbf{E}^i(\mathbf{r}) + k_o^2(\epsilon_r - 1) \mathcal{PV}_\delta \int_{V_s} \overline{\overline{G}}(\mathbf{r} - \mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dV'$$

- Interpretation: Total field $-$ field from current in V_δ = field in the cavity V_δ = incident field + field from polarization current outside the cavity

- In numerical analysis the cell of discretization can be taken as V_δ.

Numerical analysis

- Simple numerical computation:
  - Discretize the scatterer to cells $\sum \Delta V_i,\ i = 1 \cdots N$.
  - Approximate field $\mathbf{E}(\mathbf{r})$ in each cell by constant value at the center of the cell $\mathbf{E}(\mathbf{r}_i)$.
  - Solve linear system of equations for $i = 1 \cdots N$

$$(\overline{\overline{I}}+(\epsilon_r-1)\overline{\overline{L}})\cdot\mathbf{E}(\mathbf{r}_i)-k_o^2(\epsilon_r-1)\sum_{j\neq i}\overline{\overline{G}}(\mathbf{r}_i-\mathbf{r}_j)\cdot\mathbf{E}(\mathbf{r}_j)\Delta V_j = -\mathbf{E}^i(\mathbf{r}_i)$$

  - First term: diagonal element of matrix, principal-value integral: off-diagonal terms

- More efficient computation through moment methods with field approximated by suitable basis functions in cells

Field from source in complex space

- Define: distance from complex point $\mathbf{r}' = \mathbf{r}'_{re} + j\mathbf{r}'_{im}$

$$D = \sqrt{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')} \quad \text{(no conjugation!)}$$

- Green function $G(D) = e^{-jkD}/4\pi D$ satisfies Helmholtz equation (can be checked)

$$(\nabla^2 + k^2)G(D) = 0, \quad D \neq 0$$

- Singularity of Green function at $D = 0$. What does it mean?

$$D^2 = (\mathbf{r}-\mathbf{r}'_{re})^2 - 2j(\mathbf{r}-\mathbf{r}'_{re})\cdot\mathbf{r}'_{im} - \mathbf{r}'^2_{im} = 0$$

$$\Rightarrow \quad |\mathbf{r}-\mathbf{r}'_{re}| = |\mathbf{r}'_{im}|, \quad (\mathbf{r}-\mathbf{r}'_{re})\perp \mathbf{r}'_{im}$$

- Singularity circle: center at $\mathbf{r} = \mathbf{r}'_{re}$, plane $\perp$ $\mathbf{r}'_{im}$, radius $|\mathbf{r}'_{im}|$. Reduces to a point $\mathbf{r} = \mathbf{r}'_{re}$ for $\mathbf{r}'_{im} \to 0$.

- Source of $G(D) = -\delta(\mathbf{r}-\mathbf{r}')$, delta function at complex point?

Point source in complex space

- Delta function of complex variable z: limit of the Gaussian function

$$\delta_\kappa(z) = \sqrt{\kappa/\pi}\, e^{-\kappa z^2}, \quad \kappa \to \infty$$

- Denote $z = x + jy$, expand

$$\delta_\kappa(x,y) = \sqrt{\kappa/\pi}\, e^{-\kappa(x^2-y^2)}\, e^{-2j\kappa xy}$$

$$e^{-2j\kappa xy} = \cos(2\kappa xy) - j\sin(2\kappa xy) \quad \text{oscillating function}$$

- When $\kappa \to \infty$, vanishes in region $|x| > |y|$. Oscillates wildly with infinite amplitude in region $|x| < |y|$.

- Corresponds to ordinary delta function in integration if path can be changed to one along the real axis through the origin:

$$\int_a^b f(z)dz = f(z), \quad \Re\{a\} < -|\Im\{a\}|, \quad |\Im\{b\}| < \Re\{b\}$$

Double-valued Green function 1

- Complex distance function between points $\mathbf{r}$ and $\mathbf{r}' = j\mathbf{u}_z b$, $b > 0$

$$D(\mathbf{r}) = \sqrt{(\mathbf{r} - j\mathbf{r}') \cdot (\mathbf{r} - j\mathbf{r}')} = \sqrt{\rho^2 + (z - jb)^2}$$

- Assume $\mathbf{r}$ circulates a path in xz plane of radius $q \ll b$ around and through the circle of singularity:

$$\mathbf{r} = \mathbf{u}_x b + q(\mathbf{u}_x \cos\psi + \mathbf{u}_z \sin\psi), \qquad \psi = 0 \to 2\pi$$

$$D^2 = [b(\mathbf{u}_x - j\mathbf{u}_z) + q(\mathbf{u}_x \cos\psi + \mathbf{u}_z \sin\psi)]^2 \approx 2bqe^{-j\psi}$$

- The distance becomes two-valued function of geometric angle ψ

$$D \to \sqrt{2bq}\ e^{-j\psi/2}$$

- For full circulation $\psi = 0 \to 2\pi$, $D \to -D$ nonuniqueness of $\mathbf{D}(\mathbf{r})$!

Double-valued Green function 2

- Uniqueness of $D(\mathbf{r})$ and $G(D(\mathbf{r}))$ by defining two Riemann spaces glued together at the circle of singularity

$$D = \sqrt{(\mathbf{r} + j\mathbf{u}_z b)\cdot(\mathbf{r} + j\mathbf{u}_z b)} = \sqrt{r^2 - b^2 + 2jbz}$$

- Crossing branch-cut surface bounded by the singularity circle changes branch of distance function
- Two obvious choices for branch cut surfaces: (1) disk $z = 0,\ \rho < b \ \Rightarrow\ D$ imaginary and (2) holey plane $z = 0,\ \rho > b \ \Rightarrow\ D$ real
- Branch-cut surface (1) = disk corresponds to $\Re\{D\} = 0$, separates outgoing ($\Re\{D\} > 0$) and ingoing ($\Re\{D\} < 0$) solutions in the function e^{-jk_oD}
- Branch-cut surface (2) = holey plane corresponds to $\Im\{D\} = 0$, separates solutions propagating along $\mathbf{u}_z$ and $-\mathbf{u}_z$ (Gaussian beam-type of solutions)

Gaussian beam (G.A. Deschamps, 1971)

- Field from point source in complex space ≈ Gaussian beam close to z axis. Assume $\mathbf{r}' = -j\mathbf{u}_z b,\ \ \mathbf{r} = \boldsymbol{\rho} + \mathbf{u}_z z,\ \ \rho \ll z,\ \ z > 0$

$$D = \sqrt{(\mathbf{r} + j\mathbf{u}_z b)\cdot(\mathbf{r} + j\mathbf{u}_z b)} = \sqrt{\rho^2 + (z + jb)^2}$$

$$D \approx z + jb + \frac{\rho^2}{2(z + jb)} = z + jb + \frac{\rho^2}{2(z^2 + b^2)}(z - jb)$$

$$e^{-jk_o D} \approx e^{k_o b}\ e^{-jk_o z}\ \exp\left(-jk_o \frac{\rho^2 z}{2(z^2 + b^2)}\right)\ \exp\left(\frac{-\rho^2 b}{2(z^2 + b^2)}\right)$$

- Interpretation of terms: constant, plane-wave term, deviation from equi-phase plane, Gaussian field distribution for $z > 0$ when $b > 0$: Gaussian decay outwards from z axis.

- Complex point source at $\mathbf{r}' = -j\mathbf{u}_z b,\ b > 0$ creates Gaussian beam propagating in $+\mathbf{u}_z$ direction.

Field from dipole in complex space

- Field from dipole $\mathbf{J}(\mathbf{r}) = \mathbf{u}_x IL\delta(\mathbf{r} + j\mathbf{u}_z b), \quad b > 0$

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu_o IL(\mathbf{u}_x + \frac{1}{k_o^2}\mathbf{u}_x \cdot \nabla\nabla)\frac{e^{-jk_o D}}{4\pi D}$$

- Far field approximation $D = \sqrt{r^2 + 2jzb - b^2} \approx r + jb\cos\theta$

$$\mathbf{E}(\mathbf{r}) \approx -j\omega\mu_o IL\mathbf{u}_x \cdot (\overline{\overline{I}} - \mathbf{u}_r\mathbf{u}_r)\frac{e^{-jk_o r}}{4\pi r}e^{k_o b\cos\theta}, \quad \theta < \pi/2$$

$$\frac{|\mathbf{E}(\theta)|}{|\mathbf{E}(0)|} \approx \cos\theta e^{-k_o b(1-\cos\theta)} \;\; (xz \text{ plane}) \quad \approx e^{-k_o b(1-\cos\theta)} \;\; (yz \text{ plane})$$

- Directivity increases for increasing b. When $k_o b$ large, in both planes $\theta_{3dB} \approx \sqrt{2\ln 2/k_o b}$.

- Simple way to represent directive sources with sources in complex space giving Gaussian beam expansion for radiation fields.

Problems

15 Apply the volume integral equation approach to scattering from two small dielectric scatterers at the points $\mathbf{r}_1, \mathbf{r}_2$, with respective constant permittivities $\epsilon_{r1}, \epsilon_{r2}$. Assume that the contracting surfaces S_δ coincide with the boundaries of the scatterers and their distance is much larger than the sizes of the scatterers. Find the expression for the field inside each scatterer $\mathbf{E}(\mathbf{r}_1), \mathbf{E}(\mathbf{r}_2)$ when the incident field is $\mathbf{E}^i(\mathbf{r})$.

16 Find the surface of constant amplitude for a scalar field arising from a point source at the complex point $\mathbf{r}' = -j\mathbf{u}_z b,\ b > 0$. What is the surface for the amplitude corresponding to $1/e$ times the maximum amplitude of the beam?

S-96.510 Advanced Field Theory
9. Plane waves

I.V.Lindell

Plane-wave fields

- Plane wave = fields with exponential dependence on $\mathbf{r}$

$$\begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} e^{-j\mathbf{k}\cdot\mathbf{r}},$$

- Maxwell equations with $\nabla \to -j\mathbf{k}$ become (sources in infinity)

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{k} \times \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + \omega \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

- In six-vector representation

$$\mathsf{e}(\mathbf{r}) = \mathsf{e} e^{-j\mathbf{k}\cdot\mathbf{r}}, \quad J\mathbf{k} \times \mathsf{e} + \omega \mathsf{M} \cdot \mathsf{e} = 0$$

- Eigenvalue equation

$$(J\mathbf{k} \times \overline{\overline{I}} + \omega \mathsf{M}) \cdot \mathsf{e} = 0$$

- Eigenvalue parameter can be chosen freely: ω, component of $\mathbf{k}$ etc.

Dispersion equation 1

- Dispersion equation = equation for $\mathbf{k}$

$$\det(J\mathbf{k}\times\overline{\overline{I}}+\omega\mathsf{M})=\det\begin{pmatrix}\omega\overline{\overline{\epsilon}} & \omega\overline{\overline{\xi}}+\mathbf{k}\times\overline{\overline{I}}\\ \omega\overline{\overline{\zeta}}-\mathbf{k}\times\overline{\overline{I}} & \omega\overline{\overline{\mu}}\end{pmatrix}=0$$

- Eigenvalue equations for $\mathbf{H}$ and $\mathbf{E}$:

$$\overline{\overline{D}}_e(\mathbf{k})\cdot\mathbf{E}=0,\quad \overline{\overline{D}}_m(\mathbf{k})\cdot\mathbf{H}=0,$$

$$\overline{\overline{D}}_e(\mathbf{k})=\overline{\overline{H}}_e(-j\mathbf{k})=\omega^2\overline{\overline{\epsilon}}-(\omega\overline{\overline{\xi}}+\mathbf{k}\times\overline{\overline{I}})\cdot\overline{\overline{\mu}}^{-1}\cdot(\omega\overline{\overline{\zeta}}-\mathbf{k}\times\overline{\overline{I}}),$$

$$\overline{\overline{D}}_m(\mathbf{k})=\overline{\overline{H}}_m(-j\mathbf{k})=\omega^2\overline{\overline{\mu}}-(\omega\overline{\overline{\zeta}}-\mathbf{k}\times\overline{\overline{I}})\cdot\overline{\overline{\epsilon}}^{-1}\cdot(\omega\overline{\overline{\xi}}+\mathbf{k}\times\overline{\overline{I}}).$$

- Dispersion equations in 3-dyadic form

$$\det\overline{\overline{D}}_e(\mathbf{k})=0,\quad \det\overline{\overline{D}}_m(\mathbf{k})=0$$

- Same equation because $\det(\overline{\overline{I}}+\overline{\overline{A}}\cdot\overline{\overline{B}})=\det(\overline{\overline{I}}+\overline{\overline{B}}\cdot\overline{\overline{A}})$

Dispersion equation 2

- Order of the dispersion equation?

$$\mathrm{det}\overline{\overline{D}}_e(\mathbf{k}) = \omega^6 \mathrm{det}\overline{\overline{\epsilon}} - \omega^4 \overline{\overline{\epsilon}}^{(2)} : [(\omega\overline{\overline{\xi}} + \mathbf{k}\times\overline{\overline{I}})\cdot\overline{\overline{\mu}}^{-1}\cdot(\omega\overline{\overline{\zeta}} - \mathbf{k}\times\overline{\overline{I}})]$$

$$+\omega^2\overline{\overline{\epsilon}} : [(\omega\overline{\overline{\xi}} + \mathbf{k}\times\overline{\overline{I}})^{(2)}\cdot\overline{\overline{\mu}}^{(-2)}\cdot(\omega\overline{\overline{\zeta}} - \mathbf{k}\times\overline{\overline{I}})^{(2)}]$$

$$-\frac{1}{\mathrm{det}\overline{\overline{\mu}}}\mathrm{det}(\omega\overline{\overline{\xi}} + \mathbf{k}\times\overline{\overline{I}})\mathrm{det}(\omega\overline{\overline{\zeta}} - \mathbf{k}\times\overline{\overline{I}}) = 0$$

- To find sixth-order terms:

$$\mathrm{det}(\omega\overline{\overline{\xi}} + \mathbf{k}\times\overline{\overline{I}}) = \omega^3\mathrm{det}\overline{\overline{\xi}} + \omega^2\overline{\overline{\xi}}^{(2)} : (\mathbf{k}\times\overline{\overline{I}}) + \omega\overline{\overline{\xi}} : \mathbf{kk}$$

$$\mathrm{det}(\omega\overline{\overline{\zeta}} - \mathbf{k}\times\overline{\overline{I}}) = \omega^3\mathrm{det}\overline{\overline{\zeta}} - \omega^2\overline{\overline{\zeta}}^{(2)} : (\mathbf{k}\times\overline{\overline{I}}) + \omega\overline{\overline{\zeta}} : \mathbf{kk}$$

- Sixth-order and fifth-order terms absent ⇒ fourth-order equation! Dispersion equation very complicated for the general medium.

Solution of eigenwaves

- Assume $\mathbf{k} = k\mathbf{u}$, real unit vector $\mathbf{u}$, dispersion equation $k = k(\mathbf{u})$
- Defines 4th order $\mathbf{k}$-vector surface (dispersion surface) corresponding to eigensolutions (possible plane waves). k may be complex.
- In some media splits to two 2nd order surfaces (complex quadrics)
- Procedure for solution:
  - Roots $k = k_i(\mathbf{u})$ from 4th order equation $\det\overline{\overline{D}}_e(k\mathbf{u}) = 0$
  - Corresponding eigenfields $\mathbf{E}_i$ from $\overline{\overline{D}}_e(k_i\mathbf{u}) \cdot \mathbf{E}_i = 0$:
  - If $\overline{\overline{D}}_e(k_i\mathbf{u})^{(2)} \neq 0$ (k_i single eigenvalue) $\mathbf{E}_i = \mathbf{a} \cdot \overline{\overline{D}}_e^{(2)}(k_i\mathbf{u})$
  - If $\overline{\overline{D}}_e(k_i\mathbf{u})^{(2)} = 0$, ($k_i$ double eigenvalue) $\mathbf{E}_i \perp \mathbf{a} \cdot \overline{\overline{D}}_e(k_i\mathbf{u})$
  - Vector $\mathbf{a}$ chosen so that result $\neq 0$
- General plane-wave in direction $\mathbf{u}$ = sum of eigenwaves $\sum \mathbf{E}_i e^{-jk_i\mathbf{u}\cdot\mathbf{r}}$

Isotropic medium, eigenvalues

- Isotropic medium (ϵ_o, μ_o). Dispersion dyadics

$$\mu_o\overline{\overline{D}}_e(\mathbf{k}) = \epsilon_o\overline{\overline{D}}_m(\mathbf{k}) = \overline{\overline{D}}(\mathbf{k}) = k_o^2\overline{\overline{I}} + (\mathbf{k}\times\overline{\overline{I}})^2$$

$$\overline{\overline{D}}(\mathbf{k}) = \overline{\overline{D}}_+(\mathbf{k})\cdot\overline{\overline{D}}_-(\mathbf{k}), \quad \overline{\overline{D}}_\pm(\mathbf{k}) = \mathbf{k}\times\overline{\overline{I}} \mp jk_o\overline{\overline{I}}$$

- Dispersion equation splits in two 2nd order equations:

$$\det\overline{\overline{D}}_\pm(\mathbf{k}) = \mp k_o(\mathbf{k}\cdot\mathbf{k} - k_o^2) = 0 \qquad \mathbf{k}_\pm = k_o\mathbf{u}$$

- Double eigenvalue: $k_+ = k_- = k_o$.
- Dispersion surfaces: two surfaces coincide to a single sphere of radius k_o

Isotropic medium, eigenpolarizations

- Eigenpolarizations satisfy

$$\overline{\overline{D}}(\mathbf{k}_\pm)\cdot\mathbf{E}_\pm = k_o(\mathbf{u}\times\mathbf{E}_\pm \mp j\mathbf{E}_\pm) = 0 \quad \Rightarrow \quad \mathbf{E}_\pm = \mp j\mathbf{u}\times\mathbf{E}_\pm$$

- Circularly polarized eigenvectors: $\mathbf{E}_\pm\cdot\mathbf{E}_\pm = 0$, $\mathbf{u}\cdot\mathbf{E}_\pm = 0$. Helicity vector gives the handedness:

$$\mathbf{p}(\mathbf{E}_\pm) = \frac{\mathbf{E}_\pm\times\mathbf{E}_\pm^*}{j\mathbf{E}_\pm\cdot\mathbf{E}_\pm^*} = \mp\frac{j(\mathbf{u}\times\mathbf{E}_\pm)\times\mathbf{E}_\pm^*}{j\mathbf{E}_\pm\cdot\mathbf{E}_\pm^*} = \pm\mathbf{u}$$

- $\mathbf{E}_+$ right-hand, $\mathbf{E}_-$ left-hand circularly polarized
- All transverse polarizations possible because eigenwaves have same eigenvalues $k_\pm = k_o$. General plane wave can be expressed as linear combination

$$\mathbf{E}(\mathbf{r}) = (\alpha\mathbf{E}_+ + \beta\mathbf{E}_-)e^{-jk_o\mathbf{u}\cdot\mathbf{r}}$$

Bi-isotropic medium, eigenvalues

- Dispersion dyadics

  $\mu\overline{\overline{D}}_e(\mathbf{k}) = \epsilon\overline{\overline{D}}_m(\mathbf{k}) = \overline{\overline{D}}(\mathbf{k}) = \omega^2\mu\epsilon\overline{\overline{I}} - (\omega\xi\overline{\overline{I}} + \mathbf{k}\times\overline{\overline{I}})\cdot(\omega\zeta\overline{\overline{I}} - \mathbf{k}\times\overline{\overline{I}})$

- Denote $K = \omega\sqrt{\mu\epsilon}$, $\omega\xi = K(\sin\theta - j\kappa_r)$, $\omega\zeta = K(\sin\theta + j\kappa_r)$,

  $$\overline{\overline{D}}(\mathbf{k}) = K^2\overline{\overline{I}} - [K\sin\theta\overline{\overline{I}} + (\mathbf{k}\times\overline{\overline{I}} - j\kappa_r K\overline{\overline{I}})]\cdot[K\sin\theta\overline{\overline{I}} - (\mathbf{k}\times\overline{\overline{I}} - j\kappa_r K\overline{\overline{I}})]$$

  $$= K^2\cos^2\theta\overline{\overline{I}} + (\mathbf{k}\times\overline{\overline{I}} - j\kappa_r K\overline{\overline{I}})^2$$

  $$= [\mathbf{k}\times\overline{\overline{I}} + jK(\cos\theta - \kappa_r)\overline{\overline{I}}]\cdot[\mathbf{k}\times\overline{\overline{I}} - jK(\cos\theta + \kappa_r)\overline{\overline{I}}]$$

  $$= \overline{\overline{D}}_-(\mathbf{k})\cdot\overline{\overline{D}}_+(\mathbf{k}) = \overline{\overline{D}}_+(\mathbf{k})\cdot\overline{\overline{D}}_-(\mathbf{k}), \qquad \overline{\overline{D}}_\pm(\mathbf{k}) = \mathbf{k}\times\overline{\overline{I}} \mp jk_\pm\overline{\overline{I}}.$$

- Dispersion equation splits in two equations:

  $\det\overline{\overline{D}}_\pm(\mathbf{k}) = \mp jk_\pm(\mathbf{k}\cdot\mathbf{k} - k_\pm^2) = 0 \quad \mathbf{k}_\pm = \mathbf{u}k_\pm = \mathbf{u}\omega\sqrt{\mu\epsilon}(\cos\theta \pm \kappa_r)$

- $\mathbf{k}$-vector surface = two spheres of radii k_+, k_-.
  The spheres coincide when no chirality: $\kappa_r = 0$, $\Rightarrow$ $k_+ = k_-$.

Bi-isotropic medium, eigenvectors

- Eigenvectors satisfy $\overline{\overline{D}}_\pm(\mathbf{k}_\pm)\cdot\mathbf{E}_\pm = 0$

$$\overline{\overline{D}}_\pm^{(2)}(\mathbf{u}k_\pm) = (k_\pm\mathbf{u}\times\overline{\overline{I}} \mp jk_\pm\overline{\overline{I}})^{(2)} = -k_\pm^2(\overline{\overline{I}}_t \pm j\mathbf{u}\times\overline{\overline{I}})$$

$$= -k_\pm^2[\mathbf{vv}+\mathbf{uu}{}^\times_\times\mathbf{vv}\pm j(\mathbf{u}\times\mathbf{vv}-\mathbf{vu}\times\mathbf{v})] = -k_\pm^2(\mathbf{v}\pm j\mathbf{u}\times\mathbf{v})(\mathbf{v}\mp j\mathbf{u}\times\mathbf{v})$$

- Here $\mathbf{v}$ is any transverse unit vector: $\mathbf{v}\cdot\mathbf{v} = 1, \quad \mathbf{v}\cdot\mathbf{u} = 0.$
- Eigenvectors of the form $\mathbf{E}_\pm = \mathbf{a}\cdot\overline{\overline{D}}_\pm(\mathbf{k}_\pm)^{(2)} = \alpha(\mathbf{v}\mp j\mathbf{u}\times\mathbf{v})$
- Helicity vectors of eigenvectors:

$$\mathbf{p}(\mathbf{E}_\pm) = \frac{\mathbf{E}_\pm\times\mathbf{E}_\pm^*}{j\mathbf{E}_\pm\cdot\mathbf{E}_\pm^*} = \pm\mathbf{v}\times(\mathbf{u}\times\mathbf{v}) = \pm\mathbf{u},$$

- $\mathbf{E}_+$ right-hand, $\mathbf{E}_-$ left-hand circularly polarized

Self-dual medium, factorization

- Self-dual medium = generalization of bi-isotropic medium
  again defining $K = \omega\sqrt{\mu\epsilon}$

$$\omega \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \omega\epsilon & K\sin\theta \\ K\sin\theta & \omega\mu \end{pmatrix} \overline{\overline{\alpha}} + \begin{pmatrix} 0 & -jK\overline{\overline{\kappa}}_r \\ jK\overline{\overline{\kappa}}_r & 0 \end{pmatrix}$$

$$\mu\overline{\overline{D}}_e(\mathbf{k}) = K^2\overline{\overline{\alpha}} + [\mathbf{k}\times\overline{\overline{I}} + K(\sin\theta\overline{\overline{\alpha}} - j\overline{\overline{\kappa}}_r)]\cdot\overline{\overline{\alpha}}^{-1}\cdot[\mathbf{k}\times\overline{\overline{I}} - K(\sin\theta\overline{\overline{\alpha}} + j\overline{\overline{\kappa}}_r)]$$

- $\overline{\overline{D}}_e(\mathbf{k}) = \overline{\overline{H}}_e(-j\mathbf{k})$ is factorizable like the Helmholtz operator $\overline{\overline{H}}_e(\nabla)$

$$\overline{\overline{D}}_e(\mathbf{k}) = \overline{\overline{D}}_+(\mathbf{k})\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{D}}_-(\mathbf{k}) = \overline{\overline{D}}_-(\mathbf{k})\cdot\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{D}}_+(\mathbf{k})$$

$$\overline{\overline{D}}_\pm(\mathbf{k}) = \mathbf{k}\times\overline{\overline{I}} \mp jK\overline{\overline{\tau}}_\pm, \quad \overline{\overline{\tau}}_\pm = \cos\theta\,\overline{\overline{\alpha}} \pm \overline{\overline{\kappa}}_r$$

- Eigenvalue equation splits to two equations:

$$\overline{\overline{D}}_\pm(\mathbf{k})\cdot\mathbf{E}_\pm = (\mathbf{k}\times\overline{\overline{I}} \mp jK\overline{\overline{\tau}}_\pm)\cdot\mathbf{E}_\pm = 0$$

Self-dual medium, eigenvalues

- Write $\overline{\overline{\tau}}_\pm = \overline{\overline{S}}_\pm + \mathbf{a}_\pm \times \overline{\overline{I}}$ in symmetric and antisymmetric parts

$$\overline{\overline{D}}_\pm(\mathbf{k}) \cdot \mathbf{E}_\pm = (\mathbf{k} \mp jK\mathbf{a}_\pm) \times \mathbf{E}_\pm \mp jK\overline{\overline{S}}_\pm \cdot \mathbf{E}_\pm = 0$$

- Affine transformation: multiply by $\overline{\overline{S}}_\pm^{-1/2} = (\overline{\overline{S}}_\pm^{1/2})^{(2)}/\sqrt{\mathrm{det}\overline{\overline{S}}_\pm}$

$$[\overline{\overline{S}}_\pm^{1/2} \cdot (\mathbf{k} \mp jK\mathbf{a}_\pm)/\sqrt{\mathrm{det}\overline{\overline{S}}_\pm}\ ] \times (\overline{\overline{S}}_\pm^{1/2} \cdot \mathbf{E}_\pm) = \pm jK(\overline{\overline{S}}_\pm^{1/2} \cdot \mathbf{E}_\pm)$$

- This is of the form $\mathbf{k}'_\pm \times \mathbf{E}'_\pm = \pm jK\mathbf{E}'_\pm$.

$$\mathbf{k}'_\pm \cdot \mathbf{k}'_\pm = K^2, \quad \Rightarrow \quad \overline{\overline{S}}_\pm : (\mathbf{k}_\pm \mp jK\mathbf{a}_\pm)(\mathbf{k}_\pm \mp jK\mathbf{a}_\pm) = K^2\mathrm{det}\overline{\overline{S}}_\pm$$

- Second order equations for $\mathbf{k}_\pm = \mathbf{u}k_\pm$ can be solved for $k_\pm(\mathbf{u})$.
- Dispersion surfaces spheroids if $\overline{\overline{S}}_\pm$ positive definite, decentered by $\pm jK\mathbf{a}_\pm$ (real vectors for lossless medium).

Self-dual medium, eigenvectors

- Equation for the eigenvectors $\mathbf{k}'_\pm \times \mathbf{E}'_\pm = \pm jK\mathbf{E}'_\pm$ imply $\mathbf{k}'_\pm \cdot \mathbf{E}'_\pm = 0$, $\mathbf{E}'_\pm \cdot \mathbf{E}'_\pm = 0$, $\mathbf{k}'_\pm \cdot \mathbf{k}'_\pm = K^2$

- $\Rightarrow$ $\mathbf{E}'_\pm$ are circularly polarized orthogonal to $\mathbf{k}'_\pm$ with general form

$$\mathbf{E}'_\pm = \mathbf{v}_\pm \mp j\frac{\mathbf{k}'_\pm}{K} \times \mathbf{v}_\pm, \qquad \mathbf{v}_\pm \cdot \mathbf{k}'_\pm = 0$$

- Expressions can be checked by inserting in $\mathbf{k}'_\pm \times \mathbf{E}'_\pm$

- Substituting $\mathbf{k}'_\pm$ and $\mathbf{E}'_\pm = \overline{\overline{S}}_\pm^{1/2} \cdot \mathbf{E}_\pm$, the eigenvectors become

$$\mathbf{E}_\pm = \overline{\overline{S}}_\pm^{-1/2} \cdot \left(\mathbf{v}_\pm \mp j\frac{\overline{\overline{S}}_\pm^{1/2} \cdot (\mathbf{k}_\pm \mp jK\mathbf{a}_\pm)}{K\sqrt{\mathrm{det}\overline{\overline{S}}_\pm}} \times \mathbf{v}_\pm \right)$$

- Eigenvalues $\mathbf{k}_\pm = \mathbf{u}k_\pm(\mathbf{u})$ must be inserted. Vectors $\mathbf{v}_\pm$ can be expressed as $\mathbf{k}'_\pm \times \mathbf{w}$.

General anisotropic medium

- For an anisotropic medium ($\overline{\overline{\xi}} = \overline{\overline{\zeta}} = 0$) the eigenvalue equation is

$$\overline{\overline{D}}_e(\mathbf{k}) \cdot \mathbf{E} = (\omega^2 \overline{\overline{\epsilon}} - \overline{\overline{\mu}}^{-1} {}^{\times}_{\times} \mathbf{kk}) \cdot \mathbf{E} = 0$$

- The dispersion equation for $\mathbf{k}$ can be expanded as

$$\det\overline{\overline{D}}_e(\mathbf{k}) = \omega^6 \det\overline{\overline{\epsilon}} - \omega^4 (\overline{\overline{\epsilon}}^{(2)} {}^{\times}_{\times} \overline{\overline{\mu}}^{-1}) : \mathbf{kk} + \frac{\omega^2}{\det\overline{\overline{\mu}}} (\overline{\overline{\epsilon}} : \mathbf{kk})(\overline{\overline{\mu}} : \mathbf{kk}) = 0$$

- Biquadratic equation for $k(\mathbf{u})$! If $\mathbf{k}$ is a solution, also $-\mathbf{k}$ is a solution $\Rightarrow$ symmetric 4th order dispersion surface. General solution:

$$\frac{k_\pm^2(\mathbf{u})}{\omega^2} = \frac{1}{2} \left(\frac{\overline{\overline{\epsilon}}^{(2)} {}^{\times}_{\times} \overline{\overline{\mu}}^{(2)T} : \mathbf{uu}}{(\overline{\overline{\epsilon}} : \mathbf{uu})(\overline{\overline{\mu}} : \mathbf{uu})} \pm \sqrt{\frac{(\overline{\overline{\epsilon}}^{(2)} {}^{\times}_{\times} \overline{\overline{\mu}}^{(2)T} : \mathbf{uu})^2}{(\overline{\overline{\epsilon}} : \mathbf{uu})^2 (\overline{\overline{\mu}} : \mathbf{uu})^2} - \frac{4 \det\overline{\overline{\mu}} \det\overline{\overline{\epsilon}}}{(\overline{\overline{\epsilon}} : \mathbf{uu})(\overline{\overline{\mu}} : \mathbf{uu})}} \right)$$

- Eigenvectors through the previous rule.

Uniaxially anisotropic medium: TE/TM waves

- Uniaxially anisotropic medium: $\overline{\overline{\epsilon}}, \overline{\overline{\mu}}$ of the form $\overline{\overline{\alpha}} = \alpha_z \mathbf{u}_z \mathbf{u}_z + \alpha_t \overline{\overline{I}}_t$
- Polarization condition directly from Maxwell equations

$$-j\mathbf{k} \times \mathbf{E} = -j\omega\overline{\overline{\mu}} \cdot \mathbf{H}, \quad \Rightarrow \quad \mathbf{E} \cdot \overline{\overline{\mu}} \cdot \mathbf{H} = 0$$

$$-j\mathbf{k} \times \mathbf{H} = j\omega\overline{\overline{\epsilon}} \cdot \mathbf{E}, \quad \Rightarrow \quad \mathbf{H} \cdot \overline{\overline{\epsilon}} \cdot \mathbf{E} = 0$$

- Combine:

$$\mathbf{E} \cdot (\epsilon_t \overline{\overline{\mu}} - \mu_t \overline{\overline{\epsilon}}) \cdot \mathbf{H} = (\epsilon_t \mu_z - \mu_t \epsilon_z)(\mathbf{E} \cdot \mathbf{u}_z)(\mathbf{u}_z \cdot \mathbf{H}) = 0$$

- $\epsilon_t \mu_z - \mu_t \epsilon_z = 0 \quad \Rightarrow$ affine-isotropic medium satisfying $\epsilon_z \overline{\overline{\mu}} = \mu_z \overline{\overline{\epsilon}}$.
- $\epsilon_t \mu_z - \mu_t \epsilon_z \neq 0 \quad \Rightarrow$ Plane wave satisfies either $\mathbf{u}_z \cdot \mathbf{E} = 0$ (TE wave) or $\mathbf{u}_z \cdot \mathbf{H} = 0$ (TM wave).

Uniaxially anisotropic medium: effective media

- For TE and TM fields the original medium can be replaced by effective affine-isotropic media

$$\overline{\overline{\epsilon}}\cdot\mathbf{E}^{TE} = \epsilon_t\mathbf{E}^{TE} = \frac{\epsilon_t}{\mu_t}\overline{\overline{\mu}}\cdot\mathbf{E}^{TE}, \quad \Rightarrow \quad \overline{\overline{\epsilon}}^{TE} = \frac{\epsilon_t}{\mu_t}\overline{\overline{\mu}}$$

$$\overline{\overline{\mu}}\cdot\mathbf{H}^{TM} = \mu_t\mathbf{H}^{TM} = \frac{\mu_t}{\epsilon_t}\overline{\overline{\epsilon}}\cdot\mathbf{H}^{TM}, \quad \Rightarrow \quad \overline{\overline{\mu}}^{TM} = \frac{\mu_t}{\epsilon_t}\overline{\overline{\epsilon}}$$

- Special cases of self-dual media! ($\overline{\overline{\epsilon}}$ and $\overline{\overline{\mu}}$ multiples of same dyadic)

- Dispersion dyadics for effective media can be factorized:

$$\overline{\overline{D}}_e^{TE}(\mathbf{k})\cdot\mathbf{E}^{TE} = (\mathbf{k}\times\overline{\overline{I}}+j\omega\sqrt{\epsilon_t/\mu_t}\,\overline{\overline{\mu}})\cdot\overline{\overline{\mu}}^{-1}\cdot(\mathbf{k}\times\overline{\overline{I}}-j\omega\sqrt{\epsilon_t/\mu_t}\,\overline{\overline{\mu}})\cdot\mathbf{E}^{TE} = 0$$

$$\overline{\overline{D}}_m^{TM}(\mathbf{k})\cdot\mathbf{H}^{TM} = (\mathbf{k}\times\overline{\overline{I}}+j\omega\sqrt{\mu_t/\epsilon_t}\,\overline{\overline{\epsilon}})\cdot\overline{\overline{\epsilon}}^{-1}\cdot(\mathbf{k}\times\overline{\overline{I}}-j\omega\sqrt{\mu_t/\epsilon_t}\,\overline{\overline{\epsilon}})\cdot\mathbf{H}^{TM} = 0$$

- Previous solution procedure for self-dual media applicable.

Uniaxially anisotropic medium: eigenvalues

- Dispersion equations for TE and TM polarized waves

$$\det\overline{\overline{D}}_e^{TE}(\mathbf{k}) = \frac{\omega^2\epsilon_t}{\mu_t^3\mu_z}(\overline{\overline{\mu}} : \mathbf{kk} - \omega^2\mu_t\epsilon_t\mu_z)^2 = 0$$

$$\det\overline{\overline{D}}_m^{TM}(\mathbf{k}) = \frac{\omega^2\mu_t}{\epsilon_t^3\epsilon_z}(\overline{\overline{\epsilon}} : \mathbf{kk} - \omega^2\mu_t\epsilon_t\epsilon_z)^2 = 0$$

- Wave propagating in direction $\mathbf{u} = \mathbf{u}_z\cos\theta + \mathbf{u}_t\sin\theta$:

$$k^{TE} = \omega\sqrt{\frac{\mu_t\epsilon_t\mu_z}{\overline{\overline{\mu}} : \mathbf{uu}}} = \omega\sqrt{\frac{\mu_t\epsilon_t\mu_z}{\mu_z\cos^2\theta + \mu_t\sin^2\theta}}$$

$$k^{TM} = \omega\sqrt{\frac{\mu_t\epsilon_t\epsilon_z}{\overline{\overline{\epsilon}} : \mathbf{uu}}} = \omega\sqrt{\frac{\mu_t\epsilon_t\epsilon_z}{\epsilon_z\cos^2\theta + \epsilon_t\sin^2\theta}}$$

- Double eigenvalues for the effective media correspond to two single eigenvalues for the original medium.

Uniaxially anisotropic medium: eigenvectors

- Double eigenvalues $\Rightarrow$ linear dyadics $\overline{\overline{D}}_e^{TE}(\mathbf{k}^{TE}), \overline{\overline{D}}_m^{TM}(\mathbf{k}^{TM})$:

$$\overline{\overline{D}}_e^{TE}(\mathbf{k}^{TE}) = \frac{\omega^2\epsilon_t}{\mu_t}\overline{\overline{\mu}} - \frac{1}{\det\overline{\overline{\mu}}}[(\overline{\overline{\mu}}:\mathbf{kk})\overline{\overline{\mu}} - (\overline{\overline{\mu}}\cdot\mathbf{k})(\overline{\overline{\mu}}\cdot\mathbf{k})] = -\frac{(\overline{\overline{\mu}}\cdot\mathbf{k})(\overline{\overline{\mu}}\cdot\mathbf{k})}{\det\overline{\overline{\mu}}}$$

$$\overline{\overline{D}}_m^{TM}(\mathbf{k}^{TM}) = \frac{\omega^2\mu_t}{\epsilon_t}\overline{\overline{\epsilon}} - \frac{1}{\det\overline{\overline{\epsilon}}}[(\overline{\overline{\epsilon}}:\mathbf{kk})\overline{\overline{\epsilon}} - (\overline{\overline{\epsilon}}\cdot\mathbf{k})(\overline{\overline{\epsilon}}\cdot\mathbf{k})] = -\frac{(\overline{\overline{\epsilon}}\cdot\mathbf{k})(\overline{\overline{\epsilon}}\cdot\mathbf{k})}{\det\overline{\overline{\epsilon}}}$$

- TE and TM field vectors obtained from

$$\mathbf{E}^{TE}\perp\mathbf{u}_z, \quad \mathbf{E}^{TE}\perp\overline{\overline{\mu}}\cdot\mathbf{k} \quad \Rightarrow \quad \mathbf{E}^{TE} \sim \mathbf{u}_z\times\overline{\overline{\mu}}\cdot\mathbf{k} \sim \mathbf{u}_z\times\mathbf{u}$$

$$\mathbf{H}^{TM}\perp\mathbf{u}_z, \quad \mathbf{H}^{TM}\perp\overline{\overline{\epsilon}}\cdot\mathbf{k} \quad \Rightarrow \quad \mathbf{H}^{TM} \sim \mathbf{u}_z\times\overline{\overline{\epsilon}}\cdot\mathbf{k} \sim \mathbf{u}_z\times\mathbf{u}$$

- Also: directly from Maxwell equations
$\mathbf{k}\cdot\overline{\overline{\epsilon}}\cdot\mathbf{E}^{TE} = k\epsilon_t\mathbf{u}\cdot\mathbf{E}^{TE} = 0, \ \Rightarrow \ \mathbf{E}^{TE} \sim \mathbf{u}_z\times\mathbf{u}$
$\mathbf{k}\cdot\overline{\overline{\mu}}\cdot\mathbf{H}^{TM} = k\mu_t\mathbf{u}\cdot\mathbf{H}^{TM} = 0, \ \Rightarrow \ \mathbf{H}^{TM} \sim \mathbf{u}_z\times\mathbf{u}$

Uniaxial chiral medium

- Uniaxial chiral medium with parallel helices: $\zeta_z = -\xi_z = j\kappa_z\sqrt{\mu_o\epsilon_o}$

$$\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} = \begin{pmatrix} \epsilon_z\mathbf{u}_z\mathbf{u}_z + \epsilon_t\overline{\overline{I}}_t & \xi_z\mathbf{u}_z\mathbf{u}_z \\ \zeta_z\mathbf{u}_z\mathbf{u}_z & \mu_z\mathbf{u}_z\mathbf{u}_z + \mu_t\overline{\overline{I}}_t \end{pmatrix}$$

- Maxwell equations $\Rightarrow$ orthogonality conditions $\mathbf{E}\cdot\mathbf{B} = \mathbf{H}\cdot\mathbf{D} = 0$

$$\mathbf{E}\cdot\overline{\overline{\mu}}\cdot\mathbf{H} + \mathbf{E}\cdot\overline{\overline{\zeta}}\cdot\mathbf{E} = 0, \qquad \mathbf{H}\cdot\overline{\overline{\epsilon}}\cdot\mathbf{E} + \mathbf{H}\cdot\overline{\overline{\xi}}\cdot\mathbf{H} = 0$$

$$\mu_t\mathbf{E}_t\cdot\mathbf{H}_t + \mu_z E_z H_z + \zeta_z E_z^2 = 0, \quad \epsilon_t\mathbf{H}_t\cdot\mathbf{E}_t + \epsilon_z H_z E_z + \xi_z H_z^2 = 0$$

- Eliminate $\mathbf{E}_t\cdot\mathbf{H}_t$, solve for parameter A defined by

$$A = \frac{\mu_z H_z + \zeta_z E_z}{\mu_t H_z} = \frac{\epsilon_z E_z + \xi_z H_z}{\epsilon_t E_z}$$

$$\Rightarrow \quad A_\pm = \frac{1}{2}\left(\frac{\mu_z}{\mu_t} + \frac{\epsilon_z}{\epsilon_t}\right) \pm \sqrt{\frac{1}{4}\left(\frac{\mu_z}{\mu_t} - \frac{\epsilon_z}{\epsilon_t}\right)^2 + \frac{\xi_z\zeta_z}{\mu_t\epsilon_t}}$$

Uniaxial chiral medium: effective media

- Solutions $A_\pm$ correspond to two eigenwaves. Each has a special impedance relation between axial field components

$$E_{z\pm} = \frac{A_\pm \mu_t - \mu_z}{\zeta_z} H_{z\pm}, \quad \Leftrightarrow \quad H_{z\pm} = \frac{A_\pm \epsilon_t - \epsilon_z}{\xi_z} E_{z\pm}$$

- Nonchiral efficient media can be defined for the two eigenwaves

$$\mathbf{D}_\pm = \overline{\overline{\epsilon}} \cdot \mathbf{E}_\pm + \mathbf{u}_z \xi_z H_{z\pm} = (\epsilon_t \overline{\overline{I}}_t + \epsilon_t A_\pm \mathbf{u}_z \mathbf{u}_z) \cdot \mathbf{E}_\pm = \overline{\overline{\epsilon}}_\pm \cdot \mathbf{E}_\pm$$

$$\mathbf{B}_\pm = \overline{\overline{\mu}} \cdot \mathbf{H}_\pm + \mathbf{u}_z \zeta_z E_{z\pm} = (\mu_t \overline{\overline{I}}_t + \mu_t A_\pm \mathbf{u}_z \mathbf{u}_z) \cdot \mathbf{H}_\pm = \overline{\overline{\mu}}_\pm \cdot \mathbf{H}_\pm$$

- $\Rightarrow$ Affine-isotropic media with uniaxial medium dyadics:

$$\overline{\overline{\epsilon}}_\pm = \epsilon_t \overline{\overline{A}}_\pm, \quad \overline{\overline{\mu}}_\pm = \mu_t \overline{\overline{A}}_\pm, \qquad \overline{\overline{A}}_\pm = \overline{\overline{I}}_t + A_\pm \mathbf{u}_z \mathbf{u}_z$$

- The effective media are also of the self-dual form
- Eigenproblems in equivalent media can be solved with previous methods.

Uniaxial chiral medium: eigenvalues

- Dispersion equations for the two eigenwaves

$$\mathrm{det}\overline{\overline{D}}_{e\pm}(\mathbf{k}) = \mathrm{det}(\omega^2\epsilon_t\overline{\overline{A}}_\pm - \frac{1}{\mu_t}\overline{\overline{A}}^{-1} {}^{\times}_{\times}\mathbf{kk})$$

$$= \frac{\omega^2\epsilon_t}{\mu_t^2\mathrm{det}\overline{\overline{A}}_\pm}(\overline{\overline{A}}_\pm : \mathbf{kk} - k_t^2\mathrm{det}\overline{\overline{A}}_\pm)^2 = 0, \qquad k_t = \omega\sqrt{\mu_t\epsilon_t}$$

- Solutions for $\mathbf{k} = \mathbf{k}_\pm = \mathbf{u}k_\pm$

$$k_\pm = k_t\sqrt{\frac{\mathrm{det}\overline{\overline{A}}_\pm}{\overline{\overline{A}}_\pm : \mathbf{uu}}} = k_t\sqrt{\frac{A_\pm}{A_\pm\cos^2\theta + \sin^2\theta}} =$$

$$\sqrt{\frac{2\omega^2\epsilon_t\mu_t(\epsilon_z\mu_z - \xi_z\zeta_z)}{2\cos^2\theta(\epsilon_z\mu_z - \xi_z\zeta_z) + \sin^2\theta(\epsilon_t\mu_z + \mu_t\epsilon_z \mp \sqrt{(\epsilon_t\mu_z - \mu_t\epsilon_z)^2 + 4\epsilon_t\mu_t\xi_z\zeta_z})}}$$

- Uniaxial anisotropic case obtained in the limit $\xi_z \to \zeta_z \to 0$, whence $A_+ \to \mu_z/\mu_t$ (TE wave) and $A_- \to \epsilon_z/\epsilon_t$ (TM wave)

Problems

17 Show that the plane waves propagating in a bi-anisotropic medium with parameters

$$\overline{\overline{\epsilon}} = \epsilon_z \mathbf{u}_z \mathbf{u}_z + \epsilon_t \overline{\overline{I}}_t, \quad \overline{\overline{\mu}} = \mu_z \mathbf{u}_z \mathbf{u}_z + \overline{\overline{I}}_t, \quad \overline{\overline{\xi}} = \xi \mathbf{u}_z \times \overline{\overline{I}}, \quad \overline{\overline{\zeta}} = \zeta \mathbf{u}_z \times \overline{\overline{I}}$$

can be decomposed in TE and TM parts.

18 Find solutions to the wave vectors $\mathbf{k}$ for a plane wave propagating in the medium of Problem 17. Write $\xi = (\chi - j\kappa)\sqrt{\mu_o \epsilon_o}$, $\zeta = (\chi + j\kappa)\sqrt{\mu_o \epsilon_o}$ and $k_o = \omega\sqrt{\mu_o \epsilon_o}$ for simpler notation.

S-96.510 Advanced Field Theory
10. Source equivalence

I.V.Lindell

Uniqueness of sources

- Sources are uniquely defined by fields *inside* the source region

$$\begin{pmatrix} \mathbf{J}(\mathbf{r}) \\ \mathbf{M}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla\times \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix} - j\omega \begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix}$$

- Sources are not uniquely defined by fields *outside* the source region.
- Two sources g_1, g_2 are *equivalent* ($g_1 \sim g_2$) with respect to field f in region V outside the sources if their fields satisfy $f_1 = f_2$ in V.
- Difference of equivalent sources $g_1 - g_2 =$ *nonradiating* (NR) source which creates zero field in V
- Source can be replaced by an equivalent source for fields in V.
- Problem of remote sensing: how to determine sources by measuring fields outside the source region? → Additional knowledge of sources needed or to find 'simplest' of equivalent sources.

Nonradiating sources

- Nonradiating (NR) sources can be expressed in certain form. Assume linear differential equation (g is the source of the field f)

$$L(\nabla)f(\mathbf{r}) = g(\mathbf{r}) \quad \text{(+ uniqueness condition for fields)}$$

- (1) Any NR source, is obviously of the form (by definition)

$$g^{NR}(\mathbf{r}) = L(\nabla)f(\mathbf{r}), \quad f(\mathbf{r}) = 0, \ \mathbf{r} \in V$$

- (2) Conversely, any source of the form

$$g^{NR}(\mathbf{r}) = L(\nabla)h(\mathbf{r}), \quad h(\mathbf{r}) = 0 \ \mathbf{r} \in V$$

  must be nonradiating because of field uniqueness. The field $f(\mathbf{r})$ radiated by the source satisfies $L(\nabla)f(\mathbf{r}) = g^{NR}(\mathbf{r})$ whence $L(\nabla)[f(\mathbf{r})-h(\mathbf{r})] = 0$. Because of the uniqueness condition, the field in brackets is the field with no sources and, hence, it must vanish. Thus, the field $f(\mathbf{r}) = 0$ in V and $g^{NR}(\mathbf{r})$ is nonradiating.

Nonradiating electromagnetic sources

- Nonradiating electric and magnetic currents have the general form (in isotropic medium)

$$\mathbf{J}^{NR}(\mathbf{r}) = (\overline{\overline{I}}{}^{\times}_{\times}\nabla\nabla + k^2\overline{\overline{I}}) \cdot \mathbf{F}(\mathbf{r}), \qquad \mathbf{F}(\mathbf{r}) = 0,\ \mathbf{r} \in V$$

$$\mathbf{M}^{NR}(\mathbf{r}) = (\overline{\overline{I}}{}^{\times}_{\times}\nabla\nabla + k^2\overline{\overline{I}}) \cdot \mathbf{G}(\mathbf{r}), \qquad \mathbf{G}(\mathbf{r}) = 0,\ \mathbf{r} \in V$$

- Nonradiating combination of electric and magnetic currents in bi-anisotropic medium is of the form [$\mathbf{F}(\mathbf{r}) = 0$ and $\mathbf{G}(\mathbf{r}) = 0$ when $\mathbf{r} \in V$]

$$\begin{pmatrix} \mathbf{J}(\mathbf{r}) \\ \mathbf{M}(\mathbf{r}) \end{pmatrix}^{NR} = \begin{pmatrix} -j\omega\overline{\overline{\epsilon}} & \nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}} \\ -\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\zeta}} & -j\omega\overline{\overline{\mu}} \end{pmatrix} \begin{pmatrix} \mathbf{F}(\mathbf{r}) \\ \mathbf{G}(\mathbf{r}) \end{pmatrix}.$$

- Here $\mathbf{J}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ need not be nonradiating sources separately. Radiation from $\mathbf{J}$ cancels the radiation from $\mathbf{M}$ in V.

Examples of NR sources

- One-dimensional example ($\Theta(z)$ is the step function):

$$\mathbf{J}^{NR}(\mathbf{r}) = \mathbf{u}_z J[\Theta(z+a) - \Theta(z-a)]$$

- From symmetry: $\mathbf{E}(\mathbf{r}) = \mathbf{u}_z E(z), \;\Rightarrow\; \nabla \times \mathbf{E}(\mathbf{r}) = 0, \;\Rightarrow\; \mathbf{H}(\mathbf{r}) = 0$ everywhere

- $\nabla \times \mathbf{H}(\mathbf{r}) = 0 \;\Rightarrow\; \mathbf{E}(\mathbf{r}) = -\mathbf{J}(\mathbf{r})/j\omega\epsilon$, field vanishes outside current, whence $\mathbf{J}(\mathbf{r})$ does not radiate outside its support $-a < z < a$.

- Three-dimensional example: $\mathbf{J}^{NR}(\mathbf{r}) = \mathbf{u}_r J(r)\Theta(a-r)$. Radially symmetric current does not radiate fields outside its support (sphere of radius a). Symmetry $\Rightarrow$ $\mathbf{E}(\mathbf{r}) = \mathbf{u}_r E(r)$, $\nabla \times \mathbf{E}(\mathbf{r}) = 0$.

- In both cases $\nabla \times \mathbf{J}^{NR}(\mathbf{r}) = 0$ and $\mathbf{J}^{NR}(\mathbf{r}) = 0$, $\mathbf{r} \in V$, whence they are of the general NR form:

$$\mathbf{J}^{NR}(\mathbf{r}) = (\overline{\overline{I}}{}^{\times}_{\times}\nabla\nabla + k^2\overline{\overline{I}}) \cdot \mathbf{F}(\mathbf{r}), \quad \mathbf{F}(\mathbf{r}) = \mathbf{J}^{NR}(\mathbf{r})/k^2$$

Equivalence of sources

- Equivalent sources $g_1 \sim g_2$ radiate same field in V. $g_1 - g_2 \sim g^{NR}$
- Equivalence of electric and magnetic sources $\mathbf{J} \sim \mathbf{M}$ if $(\mathbf{J}, -\mathbf{M})$ is NR source of the form ($\mathbf{F}(\mathbf{r}) = 0$, $\mathbf{G}(\mathbf{r}) = 0$ when $\mathbf{r} \in V$)

$$\begin{pmatrix} \mathbf{J} \\ -\mathbf{M} \end{pmatrix} = \begin{pmatrix} -j\omega\overline{\overline{\epsilon}} & \nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}} \\ -\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\zeta}} & -j\omega\overline{\overline{\mu}} \end{pmatrix} \begin{pmatrix} \mathbf{F}(\mathbf{r}) \\ \mathbf{G}(\mathbf{r}) \end{pmatrix}.$$

- Take $\mathbf{G} = 0$ and eliminate $\mathbf{F}$ or conversely $\Rightarrow$ equivalence relations

$$\mathbf{J}(\mathbf{r}) \sim \mathbf{M} = -\frac{1}{j\omega}(\nabla \times \overline{\overline{I}} + j\omega\overline{\overline{\zeta}}) \cdot \overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J}(\mathbf{r})$$

$$\mathbf{M}(\mathbf{r}) \sim \mathbf{J} = \frac{1}{j\omega}(\nabla \times \overline{\overline{I}} - j\omega\overline{\overline{\xi}}) \cdot \overline{\overline{\mu}}^{-1} \cdot \mathbf{M}(\mathbf{r})$$

- In isotropic medium

$$\mathbf{J}(\mathbf{r}) \sim \mathbf{M}(\mathbf{r}) = -\nabla \times \mathbf{J}(\mathbf{r})/j\omega\epsilon, \quad \mathbf{M}(\mathbf{r}) \sim \mathbf{J}(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r})/j\omega\mu$$

Properties of equivalence

- Transitivity: $\mathbf{J} \sim \mathbf{M}$, $\mathbf{M} \sim \mathbf{J}' \Rightarrow \mathbf{J} \sim \mathbf{J}'$

  $$\mathbf{J} \sim \mathbf{M} = -\nabla\times\mathbf{J}/j\omega\epsilon, \;\; \mathbf{M} \sim \mathbf{J}' = \nabla\times\mathbf{M}/j\omega\mu = -\nabla\times(\nabla\times\mathbf{J})/(-k^2)$$

- Because $\mathbf{J} \sim \nabla \times (\nabla \times \mathbf{J})/k^2$, we have $-\nabla \times (\nabla \times \mathbf{J}) + k^2\mathbf{J} \sim 0$, whence this is a NR source

- A source of the form $\mathbf{J}(\mathbf{r}) = \nabla\phi(\mathbf{r})$, with $\phi(\mathbf{r}) = 0$, $\mathbf{r} \in V$ is a NR source because $\mathbf{J}(\mathbf{r}) \sim \mathbf{M}(\mathbf{r}) = -\nabla \times \mathbf{J}(\mathbf{r})/j\omega\epsilon = 0$

- A source of the form $\mathbf{J}(\mathbf{r}) = (\nabla^2 + k^2)\mathbf{F}(\mathbf{r})$, with $\mathbf{F}(\mathbf{r}) = 0$ in V, is nonradiating because adding the NR source $-\nabla(\nabla \cdot \mathbf{F}(\mathbf{r}))$ we have $\mathbf{J}(\mathbf{r}) \sim -\nabla \times (\nabla \times \mathbf{F}) + k^2\mathbf{F}$, which is a NR source

  $$\Rightarrow \qquad \mathbf{J}_1 = \nabla_t^2\mathbf{F}(\mathbf{r}) \sim \mathbf{J}_2(\mathbf{r}) = -(\partial_z^2 + k^2)\mathbf{F}(\mathbf{r})$$

Example of equivalence

- Cylindrical magnetic current $\mathbf{M}(\mathbf{r}) = \mathbf{u}_z M\Theta(a-\rho)\Theta(h^2-z^2)$
- Equivalent electric current

$$\mathbf{M}(\mathbf{r}) \sim \mathbf{J}(\mathbf{r}) = \frac{1}{j\omega\mu}\nabla\times\mathbf{M}(\mathbf{r}) = \frac{M}{j\omega\mu}\nabla\Theta(a-\rho)\times\mathbf{u}_z\Theta(h^2-z^2)$$

$$= -\mathbf{u}_\rho\times\mathbf{u}_z\frac{M}{j\omega\mu}\delta(a-\rho)\Theta(h^2-z^2) = \mathbf{u}_\varphi J_s\delta(\rho-a)\Theta(h^2-z^2)$$

- Equivalent circumferential surface current on the cylinder $\rho = a$

$$J_s = \frac{M}{j\omega\mu}$$

- Small magnetic dipole $M = I_m/\pi a^2$ length $L = 2h$ moment $I_m L$ can be replaced by a current loop of current $I = 2hJ_s = 2hM/j\omega\mu$. Relation can be written for the ratio of magnetic dipole moment and current loop moment as $I_m L/I\pi a^2 = j\omega\mu$.

Equivalent sources in electrostatics

- Static magnetic current satisfying $\nabla \cdot \mathbf{M}(\mathbf{r}) = 0$ is a source in electrostatics

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \epsilon \nabla \cdot \mathbf{E}(\mathbf{r}) = \varrho(\mathbf{r}), \qquad \nabla \times \mathbf{E}(\mathbf{r}) = -\mathbf{M}(\mathbf{r})$$

- Nonradiating combination source of the general form

$$\begin{pmatrix} \mathbf{M}(\mathbf{r}) \\ \varrho(\mathbf{r}) \end{pmatrix}^{NR} = \begin{pmatrix} \nabla \times \overline{\overline{I}} \\ \epsilon \nabla \end{pmatrix} \cdot \mathbf{F}(\mathbf{r}), \qquad \mathbf{F}(\mathbf{r}) = 0, \ \mathbf{r} \in V$$

- Equivalence: if $\mathbf{M}(\mathbf{r}) = \nabla \times \mathbf{F}(\mathbf{r}), \quad \mathbf{M}(\mathbf{r}) \sim \varrho(\mathbf{r}) = -\epsilon \nabla \cdot \mathbf{F}(\mathbf{r})$
- Static charge and magnetic current are sources of potentials

$$\nabla^2 \varphi(\mathbf{r}) = -\varrho(\mathbf{r})/\epsilon, \qquad \nabla^2 \mathbf{A}(\mathbf{r}) = \mathbf{M}(\mathbf{r}), \quad \nabla \cdot \mathbf{A}(\mathbf{r}) = 0$$

- Nonradiating sources are of the general form

$$\varrho^{NR}(\mathbf{r}) = \nabla^2 \psi(\mathbf{r}), \qquad \mathbf{M}^{NR}(\mathbf{r}) = \nabla^2 \mathbf{F}(\mathbf{r}), \quad \nabla \cdot \mathbf{F}(\mathbf{r}) = 0.$$

Multipole expansion

- Multipole = point source with structure
- Expressable as delta function series through a vector operator

$$\mathbf{J}(\mathbf{r}) = \mathbf{F}(\nabla)\delta(\mathbf{r}), \quad \mathbf{F}(\nabla) = \mathcal{P}_0 + \mathcal{P}_1 \cdot \nabla + \mathcal{P}_2 : \nabla\nabla + \cdots$$

- Multipole defined by coefficient polyadics ($n+1$-adic $\mathcal{P}_n$)
- Basic term $\mathcal{P}_0 = \mathbf{p}_0$: electric dipole $\mathcal{P}_0\delta(\mathbf{r}) = \mathbf{p}_0\delta(\mathbf{r})$
- Next term $\mathcal{P}_1 = \overline{\overline{P}}_1 = \overline{\overline{P}}_{1s} + \mathbf{p}_1 \times \overline{\overline{I}}$ (symmetric + antisymmetric parts): electric quadrupole + electric loop

$$\mathcal{P}_1 \cdot \nabla\delta(\mathbf{r}) = \overline{\overline{P}}_1 \cdot \nabla\delta(\mathbf{r}) = \overline{\overline{P}}_{1s} \cdot \nabla\delta(\mathbf{r}) + \mathbf{p}_1 \times \nabla\delta(\mathbf{r})$$

- Magnetic dipole is equivalent to an electric loop

$$\mathbf{M}(\mathbf{r}) \sim \mathbf{J}(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r})/j\omega\mu, \quad \Rightarrow \quad \mathbf{M} = -j\omega\mu\mathbf{p}_1$$

Shifting operator

- Shifting operator $e^{-\mathbf{a}\cdot\nabla}$ produces Taylor expansion

$$e^{-\mathbf{a}\cdot\nabla} = 1 - \mathbf{a}\cdot\nabla + \tfrac{(-1)^2}{2!}(\mathbf{aa}) : (\nabla\nabla) + \tfrac{(-1)^3}{3!}(\mathbf{aaa}) \,③\, (\nabla\nabla\nabla) + \cdots$$

- Taylor expansion ⇒ shifting of dipole from origin to $\mathbf{r} = \mathbf{a}$

$$e^{-\mathbf{a}\cdot\nabla} f(\mathbf{r}) = f(\mathbf{r}-\mathbf{a}) \quad \Rightarrow \quad e^{-\mathbf{a}\cdot\nabla}\mathbf{p}\delta(\mathbf{r}) \sim \mathbf{p}\delta(\mathbf{r}-\mathbf{a})$$

- Dipole at $\mathbf{r} = \mathbf{a}$ equivalent to multipole at origin:

$$\mathbf{p}\delta(\mathbf{r}-\mathbf{a}) \sim \mathbf{p}\delta(\mathbf{r}) - \mathbf{pa}\cdot\nabla\delta(\mathbf{r}) + \frac{1}{2}\mathbf{paa} : \nabla\nabla\delta(\mathbf{r}) + \cdots$$

- Same dipole moment! In addition magnetic dipole + electric quadrupole + higher order multipoles. Second-order terms:

$$-\mathbf{pa}\cdot\nabla\delta(\mathbf{r}) = \frac{1}{2}(\mathbf{p}\times\mathbf{a})\times\nabla\delta(\mathbf{r}) - \frac{1}{2}(\mathbf{pa}+\mathbf{ap})\cdot\nabla\delta(\mathbf{r})$$

- No magnetic dipole (electric current loop) if $\mathbf{a}$ parallel to $\mathbf{p}$!

Multipole of localized source

- Element of current $\mathbf{J}(\mathbf{r})$ at $\mathbf{r}'$= point source $\sim$ multipole at origin

$$\mathbf{J}(\mathbf{r}')dV'\delta(\mathbf{r}-\mathbf{r}') \sim \mathbf{J}(\mathbf{r}')dV'e^{-\mathbf{r}'\cdot\nabla}\delta(\mathbf{r})$$

- Localized source in V = integral of elements $\sim$ multipole
$\mathbf{J}(\mathbf{r}) = \int_V \mathbf{J}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')dV' \sim \int_V \mathbf{J}(\mathbf{r}')e^{-\mathbf{r}'\cdot\nabla}dV'\delta(\mathbf{r}) = \mathbf{L}(\nabla)\delta(\mathbf{r})$

- $\mathbf{L}(\nabla) = \int \mathbf{J}(\mathbf{r}')e^{-\mathbf{r}'\cdot\nabla}dV'$ operator corresponding to source $\mathbf{J}(\mathbf{r})$,
  ⓝ $= n$-dot product, convergence of series assumed

$$\mathbf{L}(\nabla) = \int_V \mathbf{J}(\mathbf{r}')e^{-\mathbf{r}'\cdot\nabla}dV' = \sum_{n=0}^{\infty} \mathcal{P}_n \; ⓝ \; \nabla\nabla\cdots\nabla,$$

$$\mathcal{P}_n = \int_V \frac{(-1)^n}{n!}\mathbf{J}(\mathbf{r}')\mathbf{r}'\mathbf{r}'\cdots\mathbf{r}'dV' \qquad n \text{ vectors } \mathbf{r}'$$

Another definition of multipole

- Vector potential from a small source in isotropic space ($\mathbf{r} \neq V$)

$$\mathbf{A}(\mathbf{r}) = \mu \int_V G(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}')dV', \qquad G(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r}$$

- Apply shifting operator to $G(\mathbf{r}-\mathbf{r}')$

$$G(\mathbf{r}-\mathbf{r}') = e^{-\mathbf{r}'\cdot\nabla}G(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}[\nabla\nabla\cdots\nabla]G(\mathbf{r}) \; ⓝ \; [\mathbf{r}'\mathbf{r}'\cdots\mathbf{r}']$$

$$\mathbf{A}(\mathbf{r}) = \mu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}[\nabla\nabla\cdots\nabla]G(\mathbf{r}) \; ⓝ \int_V [\mathbf{r}'\mathbf{r}'\cdots\mathbf{r}']\mathbf{J}(\mathbf{r}')dV'$$

$$\mathbf{A}(\mathbf{r}) = \mu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}[\nabla\nabla\cdots\nabla]G(\mathbf{r}) \; ⓝ \, \mathcal{P}_n$$

- Derivatives of the Green function for different multipole terms

Lowest-order multipole terms

- Dipole term

$$\mathcal{P}_0 = \int_V \mathbf{J}(\mathbf{r}')dV' = j\omega\mathbf{p}_e$$

- Magnetic dipole term: antisymmetric part of $\mathcal{P}_1$

$$\mathcal{P}_{1a} = \frac{1}{2}\int_V (\mathbf{J}(\mathbf{r}')\mathbf{r}' - \mathbf{r}'\mathbf{J}(\mathbf{r}'))dV' = \frac{1}{2}\int_V (\mathbf{r}' \times \mathbf{J}(\mathbf{r}'))dV' \times \overline{\overline{I}} = \mathbf{p}_m \times \overline{\overline{I}}$$

- Electric quadrupole term: symmetric part of $\mathcal{P}_1$

$$\mathcal{P}_{1s} = \frac{1}{2}\int_V (\mathbf{J}(\mathbf{r}')\mathbf{r}' + \mathbf{r}'\mathbf{J}(\mathbf{r}'))dV' = \frac{j\omega}{2}\overline{\overline{Q}}_e$$

- Usually approximations do not proceed further - enough for sufficiently concentrated sources

Location of the multipole

- Field from multipole at origin

$$\mathbf{A}(\mathbf{r}) = \mu G(\mathbf{r}) \int_V \mathbf{J}(\mathbf{r}')dV' - \mu[\nabla G(\mathbf{r})] \cdot \int_V \mathbf{r}'\mathbf{J}(\mathbf{r}')dV' + \cdots$$

- Multipole shifted to $\mathbf{r} = \mathbf{a}$:

$$\mathbf{A}(\mathbf{r}) = \mu G(\mathbf{r}-\mathbf{a}) \int_V \mathbf{J}(\mathbf{r}')dV' - \mu[\nabla G(\mathbf{r}-\mathbf{a})] \cdot \int_V (\mathbf{r}'-\mathbf{a})\mathbf{J}(\mathbf{r}')dV' + \cdots$$

$$= \mu G(\mathbf{r}-\mathbf{a})\mathbf{p}_0 - \mu[\nabla G(\mathbf{r}-\mathbf{a})] \cdot [\overline{\overline{P}}_1 - \mathbf{a}\mathbf{p}_0] + \cdots$$

- Basic term (dipole $\mathbf{p}_0$) remains the same, other terms are changed
- Best dipole approximation: choose $\mathbf{a}$ so that the second term is small $\Rightarrow$ to minimize norm of dyadic $\overline{\overline{P}}_1 - \mathbf{a}\mathbf{p}_0$ (assuming $\mathbf{p}_0 \neq 0$).

Optimal location of multipole

- Choose Erhard Schmidt norm $\|\overline{\overline{A}}\| = \sqrt{\overline{\overline{A}} : \overline{\overline{A}}^*} = \sum\sum |A_{ij}|^2$

$$[\overline{\overline{P}}_1 - \mathbf{a}\mathbf{p}_0] : [\overline{\overline{P}}_1^* - \mathbf{a}^*\mathbf{p}_0^*] = \text{minimum}$$

- Gradient in $\mathbf{a}$ space $= -[\overline{\overline{P}}_1^* - \mathbf{a}^*\mathbf{p}_0^*] \cdot \mathbf{p}_0 = 0 \Rightarrow$ solve for $\mathbf{a}^*$

$$\mathbf{a} = \frac{\overline{\overline{P}}_1 \cdot \mathbf{p}_0^*}{\mathbf{p}_0 \cdot \mathbf{p}_0^*} = \frac{\int \mathbf{r}'\mathbf{J}(\mathbf{r}')dV' \cdot \int \mathbf{J}^*(\mathbf{r}')dV'}{\int \mathbf{J}(\mathbf{r}')dV' \cdot \int \mathbf{J}^*(\mathbf{r}')dV'}, \qquad (\mathbf{p}_0 \neq 0)$$

- Example: if $\mathbf{J}(\mathbf{r}) = \mathbf{u}J(\mathbf{r})$, constant $\mathbf{u}$ (direction of current)

$$\mathbf{a} = \frac{\int \mathbf{r}' J(\mathbf{r}')dV'}{\int J(\mathbf{r}')dV'}, \qquad '\text{center of gravity}'$$

- In general $\mathbf{a}$ is a complex vector

Example

- Origocentric cubic current (side = L): $\mathbf{J}(\mathbf{r}) = \mathbf{u}_z J e^{-jkx}$,

$$\mathbf{p}_0 = \int_V \mathbf{J}(\mathbf{r}')dV' = \mathbf{u}_z J L^3 \frac{\sin\tau}{\tau}, \qquad \tau = kL/2 = \pi L/\lambda$$

$$\overline{\overline{P}}_1 = \int_V \mathbf{r}'\mathbf{J}(\mathbf{r}')dV' = \mathbf{u}_x\mathbf{u}_z J \frac{jL^3}{k^3}(\cos\tau - \sin\tau/\tau)$$

$$\mathbf{a} = \frac{\overline{\overline{P}}_1 \cdot \mathbf{p}_0^*}{\mathbf{p}_0 \cdot \mathbf{p}_0^*} = \mathbf{u}_x \frac{j}{k}(\tau\cos\tau - 1) \quad \text{imaginary vector!}$$

- For a small cube, $kL \ll 1 \Rightarrow \tau \ll 1$:

$$\tau\cot\tau - 1 = \frac{\tau\cos\tau - \sin\tau}{\sin\tau} \approx -\frac{\tau^2}{3}$$

$$\mathbf{a} \approx -\mathbf{u}_x(jkL^2/12), \quad \text{small and imaginary}$$

Radiation patterns

- Exact radiation pattern in far field obtained through integration:

$$F(\theta,\varphi) = \frac{\sin(\tau(1-\sin\theta\cos\varphi))}{\tau(1-\sin\theta\cos\varphi)}\,\frac{\sin(\tau\sin\theta\sin\varphi)}{\tau\sin\theta\sin\varphi}\,\frac{\sin(\tau\cos\theta)}{\tau\cos\theta}\,\sin\theta$$

- Compare with approximation by dipole at origin

$$F_0(\theta,\varphi) = \sin\theta, \qquad \text{no dependence on } \varphi$$

- Dipole at complex point $\mathbf{a}$:

$$F_{\mathbf{a}}(\theta,\varphi) = \sin\theta\exp[(1-\tau\cot\tau)(1-\sin\theta\cos\varphi)]$$

- Gives correct two-term approximation for small τ:

$$F(\theta,\varphi) \to F_{\mathbf{a}}(\theta,\varphi) \to \sin\theta[1 - \frac{\tau^2}{3}(1-\sin\theta\cos\varphi)]$$

- Multipole with optimized location gives one order better approximation than origocentric multipole

Extension of multipoles

- Multipole expression of a localized source $g(\mathbf{r})$ corresponds to Taylor expansion of the operator $L(\nabla)$:

$$g(\mathbf{r}) = \int_{V_g} \delta(\mathbf{r}-\mathbf{r}')g(\mathbf{r}')dV' = \int_{V_g} e^{-\mathbf{r}'\cdot\nabla} g(\mathbf{r}')dV'\delta(\mathbf{r}) = L(\nabla)\delta(\mathbf{r}),$$

$$L(\nabla) = \int_{V_g} g(\mathbf{r}')dV' - \int_{V_g} g(\mathbf{r}')\mathbf{r}'dV'\cdot\nabla + \frac{1}{2!}\int_{V_g} g(\mathbf{r}')\mathbf{r}'\mathbf{r}'dV' : \nabla\nabla\cdots$$

- Conversely, multipole can be extended to an equivalent volume source if the multipole series can be expressed in terms of an analytic operator $L(\nabla)$ as

$$\sum_{n=0}^{\infty} \mathcal{L}_n \,ⓝ\, (\nabla\cdots\nabla)\delta(\mathbf{r}) = L(\nabla)\delta(\mathbf{r}),$$

  and if this expression can be interpreted as a function of $\mathbf{r}$

Example of multipole extension

- To find the extended source of the multipole

$$g(\mathbf{r}) = \delta(\mathbf{r}) + \frac{1}{3!}(\mathbf{a}\cdot\nabla)^2\delta(\mathbf{r}) + \frac{1}{5!}(\mathbf{a}\cdot\nabla)^4\delta(\mathbf{r}) + \cdots = \frac{\sinh(\mathbf{a}\cdot\nabla)}{\mathbf{a}\cdot\nabla}\delta(\mathbf{r})$$

- **Solution:** assume for simplicity that $\mathbf{a} = \mathbf{u}_z a$ and integrate

$$g(\mathbf{r}) = \frac{1}{2a\partial_z}(e^{a\partial_z} - e^{-a\partial_z})\delta(\mathbf{r}) = \frac{1}{2a\partial_z}(\delta(z+a) - \delta(z-a))\delta(\boldsymbol{\rho})$$

$$\partial_z g(\mathbf{r}) = \frac{1}{2a}(\delta(z+a) - \delta(z-a))\delta(\boldsymbol{\rho})$$

$$g(\mathbf{r}) = \frac{1}{2a}(\Theta(z+a) - \Theta(z-a))\delta(\boldsymbol{\rho})$$

- Resulting equivalent source = line source of constant amplitude $1/2a$ between $-a < z < a$. ($\Theta(z)$ = unit step function)

Fourier transforms

- Relation between the operator expression $L(\nabla)$ and the source function $g(\mathbf{r})$ can be expressed as Fourier transformation

$$L(\nabla) = \int e^{j\mathbf{r}'\cdot j\nabla} g(\mathbf{r}')dV' = h(j\nabla)$$

- The inverse transformation is

$$g(\mathbf{r}') = \frac{1}{(2\pi)^3}\int e^{-j\mathbf{r}'\cdot\mathbf{p}} h(\mathbf{p})dV_p$$

- If the function $h(\mathbf{p}) = L(-j\mathbf{p})$ corresponding to an operator $L(\nabla)$ has an inverse function in Fourier transform tables, source function $g(\mathbf{r})$ can be found in analytic form.

Example

- For example, the previous operator gives

$$L(\nabla) = \frac{\sinh(\mathbf{a}\cdot\nabla)}{\mathbf{a}\cdot\nabla} \quad \Rightarrow \quad h(\mathbf{p}) = L(-j\mathbf{p}) = \frac{\sin(\mathbf{a}\cdot\mathbf{p})}{\mathbf{a}\cdot\mathbf{p}}$$

$$g(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{-jxp_x}dp_x \int_{-\infty}^{\infty} e^{-jyp_y}dp_y \int_{-\infty}^{\infty} e^{-jzp_z}\frac{\sin(ap_z)}{ap_z}dp_z$$

- Using known results from transform tables:

$$\int_{-\infty}^{\infty} e^{-jxp_x}dp_x = 2\pi\delta(x), \qquad \int_{-\infty}^{\infty} e^{-jzp_z}\frac{\sin ap_z}{ap_z}dp_z = \begin{pmatrix} \pi/a & |z|<a \\ 0 & |z|>a \end{pmatrix}$$

- the previous result is obtained:

$$g(\mathbf{r}) = \delta(x)\delta(y)\frac{1}{2a}(\Theta(z+a) - \Theta(z-a))$$

Problems

19 Find the multipole at the origin, equivalent to the line charge $q_i(z)$ component of the image $\varrho_i(\mathbf{r})$ of a point charge Q at $z = d$, in front of a dielectric sphere of radius a and relative permittivity ϵ_r:

$$\varrho_i(\mathbf{r}) = [-Q_i\delta(z - d_K) + q_i(z)]\delta(\boldsymbol{\rho}), \qquad q(z) = \frac{\alpha Q_i}{d_K}(z/d_K)^{\alpha-1}$$

$$Q_i = Q\frac{\epsilon_r - 1}{\epsilon_r + 1}\frac{a}{d}, \quad \alpha = 1/(\epsilon_r + 1), \quad d_K = a^2/d.$$

The image is zero outside $0 \leq z \leq d_K$. What is the best location of the multipole (no dipole term)?

20 Find the extended source equivalent to the multipole

$$g(\mathbf{r}) = \delta(\mathbf{r}) + \frac{1}{3}(\mathbf{a}\cdot\nabla)^2\delta(\mathbf{r}) + \frac{2}{45}(\mathbf{a}\cdot\nabla)^2\delta(\mathbf{r}) + \cdots = \frac{\sinh^2(\mathbf{a}\cdot\nabla)}{(\mathbf{a}\cdot\nabla)^2}\delta(\mathbf{r})$$

S-96.510 Advanced Field Theory
11. Huygens' Principle

I.V.Lindell

Truncated sources and fields

- Maxwell equations in six-vector notation

$$\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla\times\overline{\overline{I}} - j\omega\begin{pmatrix} \overline{\overline{\epsilon}} & \overline{\overline{\xi}} \\ \overline{\overline{\zeta}} & \overline{\overline{\mu}} \end{pmatrix}\right]\cdot\begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \mathbf{J}(\mathbf{r}) \\ \mathbf{M}(\mathbf{r}) \end{pmatrix}$$

$$[J\nabla\times\overline{\overline{I}} - j\omega\mathsf{M}]\cdot\mathsf{e}(\mathbf{r}) = \mathsf{j}(\mathbf{r})$$

- Huygens' source = source on a closed surface equivalent to original sources behind the surface

- Assume closed surface S dividing all space in V_1 and V_2 with respective unit normal vectors $\mathbf{n}_1 = -\mathbf{n}_2$. Define complementary pulse functions $(P_1(\mathbf{r}) + P_2(\mathbf{r}) = 1)$

$$P_1(\mathbf{r}) = 1, P_2(\mathbf{r}) = 0,\ \mathbf{r}\in V_1, \qquad P_2(\mathbf{r}) = 1, P_1(\mathbf{r}) = 0,\ \mathbf{r}\in V_2$$

- Define complementary truncated sources and fields

$$\mathsf{e}_{1,2}(\mathbf{r}) = P_{1,2}(\mathbf{r})\mathsf{e}(\mathbf{r}), \qquad \mathsf{j}_{1,2}(\mathbf{r}) = P_{1,2}(\mathbf{r})\mathsf{j}(\mathbf{r})$$

Sources of truncated fields

- Consider Maxwell equations for $\mathsf{e}_1(\mathbf{r}) = P_1(\mathbf{r})\mathsf{e}(\mathbf{r})$:

$$[J\nabla \times \overline{\overline{I}} - j\omega\mathsf{M}] \cdot \mathsf{e}_1(\mathbf{r}) =$$

$$= P_1(\mathbf{r})[J\nabla \times \overline{\overline{I}} - j\omega\mathsf{M}] \cdot \mathsf{e}(\mathbf{r}) + J\nabla P_1(\mathbf{r}) \times \mathsf{e}(\mathbf{r})$$
$$= P_1(\mathbf{r})\mathsf{j}(\mathbf{r}) + J\nabla P_1(\mathbf{r}) \times \mathsf{e}(\mathbf{r}) = \mathsf{j}_1(\mathbf{r}) + \mathsf{j}_{H1}(\mathbf{r})$$

- Source of the truncated field $\mathsf{e}_1(\mathbf{r})$ is the truncated source $\mathsf{j}_1(\mathbf{r})$ + Huygens' source $\mathsf{j}_{H1}(\mathbf{r})$

$$\mathsf{j}_{H1}(\mathbf{r}) = J\nabla P_1(\mathbf{r}) \times \mathsf{e}(\mathbf{r}) = \begin{pmatrix} \nabla P_1(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) \\ -\nabla P_1(\mathbf{r}) \times \mathbf{E}(\mathbf{r}) \end{pmatrix}$$

- Huygens' electric and magnetic current sources for e_1:

$$\mathbf{J}_{H1}(\mathbf{r}) = \nabla P_1(\mathbf{r}) \times \mathbf{H}(\mathbf{r}), \quad \mathbf{M}_{H1}(\mathbf{r}) = -\nabla P_1(\mathbf{r}) \times \mathbf{E}(\mathbf{r})$$

- Similar expressions for index 2

Huygens' sources

- Gradient of the pulse function = surface delta function (assume $f(\mathbf{r}) = 0$ at infinity)

$$\int_{V_1+V_2} f(\mathbf{r})\nabla P_1(\mathbf{r})dV = \int_{V_1+V_2} \nabla[f(\mathbf{r})P_1(\mathbf{r})]dV - \int_{V_1+V_2} [\nabla f(\mathbf{r})]P_1(\mathbf{r})dV$$

$$= -\int_{V_1} \nabla f(\mathbf{r})dV = -\oint_S \mathbf{n}_2 f(\mathbf{r})dS, \quad \Rightarrow \quad \nabla P_1(\mathbf{r}) = \mathbf{n}_1 \delta_S(\mathbf{r})$$

- Huygens' sources: surface sources on S

$$\mathbf{J}_{H1}(\mathbf{r}) = \mathbf{n}_1 \times \mathbf{H}(\mathbf{r})\delta_S(\mathbf{r}), \qquad \mathbf{M}_{H1}(\mathbf{r}) = -\mathbf{n}_1 \times \mathbf{E}(\mathbf{r})\delta_S(\mathbf{r})$$

- Huygens' principle: source $\mathsf{j}_2(\mathbf{r}) = \mathsf{j}(\mathbf{r}) - \mathsf{j}_1(\mathbf{r})$ can be replaced by $\mathsf{j}_{H1}(\mathbf{r})$ for the field in V_1.

- Huygens' principle does not depend on the medium! Can be isotropic, anisotropic, bi-anisotropic, even nonlinear.

Properties of Huygens' sources

- Because $\mathbf{n}_2 = -\mathbf{n}_1$, Huygens' sources satisfy $\mathsf{j}_{H2}(\mathbf{r}) = -\mathsf{j}_{H1}(\mathbf{r})$
- Equivalence of sources (with respect to volume in brackets)

$$\mathsf{j}_{H1}(\mathbf{r}) \sim \mathsf{j}_2(\mathbf{r}),\ (V_1), \qquad \mathsf{j}_{H2}(\mathbf{r}) \sim \mathsf{j}_1(\mathbf{r}),\ (V_2)$$

- Nonradiating sources (with respect to volume in brackets)

$$\mathsf{j}_2^{NR}(\mathbf{r}) = \mathsf{j}_{H2}(\mathbf{r}) + \mathsf{j}_2(\mathbf{r}),\ (V_1), \qquad \mathsf{j}_1^{NR}(\mathbf{r}) = \mathsf{j}_{H1}(\mathbf{r}) + \mathsf{j}_1(\mathbf{r}),\ (V_2)$$

- If all sources are in V_1, $\mathsf{j}_2(\mathbf{r}) = 0$

$$\Rightarrow \quad \mathsf{j}_{H1}(\mathbf{r}) = \text{NR source } (V_1) \quad \Rightarrow \quad \mathbf{J}_{H1}(\mathbf{r}) \sim -\mathbf{M}_{H1}(\mathbf{r})\ (V_1)$$

- Absorbing boundary can be realized by NR Huygens' source

Huygens' principle for planar S

- If S is plane, image principle simplifies Huygens' sources
- Since $\mathrm{j}_1(\mathbf{r}) + \mathrm{j}_{H1}(\mathbf{r})$ is NR (V_2), volume V_2 can be filled with any medium, for example, PEC or PMC
- PEC or PMC half space can be replaced by the corresponding image of the source $\mathrm{j}_1(\mathbf{r}) + \mathrm{j}_{H1}(\mathbf{r})$
- If $\mathrm{j}_1(\mathbf{r}) = 0$ (original source in V_2) $\Rightarrow$ $\mathrm{j}_{H1}(\mathbf{r})$ is NR source (V_2)
- In this case $\mathbf{J}_{H1}(\mathbf{r}) \sim -\mathbf{M}_{H1}(\mathbf{r})$ with respect to V_2
- From symmetry: $\mathbf{J}_{H1}(\mathbf{r}) \sim +\mathbf{M}_{H1}(\mathbf{r})$ with respect to V_1
- Huygens' source $\mathrm{j}_{H1}(\mathbf{r}) \sim 2\mathbf{J}_{H1}(\mathbf{r}) \sim 2\mathbf{M}_{H1}(\mathbf{r})$ with respect to V_1
- Simplifies problems of wave transmission through apertures in PEC plane $\Rightarrow$ only electric field in apertures needs to be solved.

How to use Huygens' principle?

- Fields from complicated sources and structures can be computed by replacing sources and structures through Huygens' sources on a surface S
- Problem: fields not known on $S \Rightarrow$ Huygens' sources not known
- Principle can be applied approximatively, two main methods:
  1. In some problems fields can be approximated on a surface S with good accuracy. For example, if S is a conducting plane with a thin slot, field in a slot can be easily approximated.
  2. In more complex cases surface integral equations on S can be formed for unknown Huygens' sources. Solution through numerical methods.
- Forming the surface integral equation will be considered through an example of scattering by an object

Scattering problem

- Assume source $\mathsf{j}(\mathbf{r}) = \mathsf{j}_1(\mathbf{r})$ in isotropic medium V_1. Scatterer V_2 is replaced by the Huygens' source $\mathsf{j}_{H1}(\mathbf{r})$ on its boundary S

- Incident field $\mathsf{e}^i(\mathbf{r})$ gives rise to scattered field $\mathsf{e}^{sc}(\mathbf{r})$:

$$\mathsf{e}^i(\mathbf{r}) = -\int_{V_1} \mathsf{G}(\mathbf{r}-\mathbf{r}')\cdot\mathsf{j}(\mathbf{r}')dV', \quad \mathsf{e}^{sc}(\mathbf{r}) = -\oint_S \mathsf{G}(\mathbf{r}-\mathbf{r}')\cdot\mathsf{j}_{H1}(\mathbf{r}')dV'$$

- Integral equation for $\mathsf{j}_{H1}(\mathbf{r})$ similar to volume integral equation?

$$\begin{pmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \mathbf{E}^i(\mathbf{r}) \\ \mathbf{H}^i(\mathbf{r}) \end{pmatrix} - \oint_S \begin{pmatrix} \overline{\overline{G}}_{ee}(\mathbf{r}-\mathbf{r}') & \overline{\overline{G}}_{em}(\mathbf{r}-\mathbf{r}') \\ \overline{\overline{G}}_{me}(\mathbf{r}-\mathbf{r}') & \overline{\overline{G}}_{mm}(\mathbf{r}-\mathbf{r}') \end{pmatrix} \cdot \begin{pmatrix} \mathbf{n}_1 \times \mathbf{H}(\mathbf{r}') \\ -\mathbf{n}_1 \times \mathbf{E}(\mathbf{r}') \end{pmatrix} dS'$$

- When $\mathbf{r} \to S$, Green dyadics become too singular for surface sources! Integral equation must be reformulated.

Singularity problem

- Principal-value decomposition is not valid for surface sources because $\overline{\overline{L}} \cdot \mathbf{J}(\mathbf{r}) = \overline{\overline{L}} \cdot \mathbf{J}_s(\mathbf{r})\delta_S(\mathbf{r})$ is infinite due to the delta function
- However: fields at the surface sources are finite!
- For example, constant time-harmonic planar surface current $\mathbf{J}(\mathbf{r}) = \mathbf{J}_s\delta(z)$ creates the plane wave

$$\mathbf{E}(z) = -\frac{\eta}{2}\mathbf{J}_s e^{-jk|z|}, \quad \mathbf{H}(z) = -\frac{1}{2}\mathrm{sgn}(z)\mathbf{u}_z \times \mathbf{J}_s e^{-jk|z|}$$

- $\mathbf{E}(z)$ is symmetric and continuous, $\mathbf{H}(z)$ is antisymmetric and discontinuous at $z = 0$, both are finite.
- For planar magnetic surface current $\mathbf{M}(\mathbf{r}) = \mathbf{M}_s\delta(z)$:

$$\mathbf{H}(z) = -\frac{1}{2\eta}\mathbf{M}_s e^{-jk|z|}, \quad \mathbf{E}(z) = \frac{1}{2}\mathrm{sgn}(z)\mathbf{u}_z \times \mathbf{M}_s e^{-jk|z|}$$

Dissolution of Green dyadic

- Extracting the most singular term $\nabla\nabla G$ of Green function.

$$\oint_S \overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}_s(\mathbf{r}')dS' = \oint_S G\mathbf{J}_s dS' + \frac{1}{k^2}\oint_S (\nabla'\nabla' G)\cdot\mathbf{J}_s dS'$$

- Partial integration on closed surface S:

$$\oint_S (\nabla'\nabla' G)\cdot\mathbf{J}_s dS' = \oint_S \nabla'\cdot(\mathbf{J}_s\nabla' G)dS' - \oint_S (\nabla'\cdot\mathbf{J}_s)\nabla' G dS'$$

- $\oint \nabla'\cdot\overline{\overline{F}}dS = 0$ on closed surface when $\overline{\overline{F}}$ is continuous

$$\oint_S \overline{\overline{G}}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}_s(\mathbf{r}')dS' = \oint_S G\mathbf{J}_s dS' - \frac{1}{k^2}\oint_S (\nabla'\cdot\mathbf{J}_s)\nabla' G dS'$$

- $\nabla G(\mathbf{r}-\mathbf{r}')$ is not too singular like $\nabla\nabla G(\mathbf{r}-\mathbf{r}')$. Drawback: differentiation of $\mathbf{J}_s(\mathbf{r}')$ on S required $\Rightarrow$ more numerical work.

Fields from surface sources

- Define: electric field function, gives electric field due to electric and magnetic surface sources on closed surface S:

$$\mathcal{E}[\mathbf{J}_s|\mathbf{M}_s]_S = -j\omega\mu \oint_S G(\mathbf{r}-\mathbf{r}')\mathbf{J}_s dS'$$

$$-\frac{1}{j\omega\epsilon}\oint_S (\nabla'\cdot\mathbf{J}_s(\mathbf{r}'))\nabla' G(\mathbf{r}-\mathbf{r}')dS' + \oint_S \nabla' G(\mathbf{r}-\mathbf{r}')\times\mathbf{M}_s(\mathbf{r}')dS'$$

- Similarly: magnetic field function

$$\mathcal{H}[\mathbf{J}_s|\mathbf{M}_s]_S = -j\omega\epsilon \oint_S G(\mathbf{r}-\mathbf{r}')\mathbf{M}_s(\mathbf{r}')dS'$$

$$-\frac{1}{j\omega\mu}\oint_S (\nabla'\cdot\mathbf{M}_s(\mathbf{r}'))\nabla' G(\mathbf{r}-\mathbf{r}')dS' - \oint_S \nabla' G(\mathbf{r}-\mathbf{r}')\times\mathbf{J}_s(\mathbf{r}')dS'$$

Fields from surface sources 2

- $\mathcal{E}[\mathbf{J}_s|\mathbf{M}_s]_S$, and $\mathcal{H}[\mathbf{J}_s|\mathbf{M}_s]_S$ give the radiated electric and magnetic fields $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ when $\mathbf{r}$ outside S.

- Questions: (1) How to compute $\mathcal{E}$ and $\mathcal{H}$ when $\mathbf{r} \in S$? (2) What is their relation to limit $\mathbf{r} \rightarrow S$ of $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ from both sides of surface?

- $\oint G\mathbf{J}_s dS'$ can be taken in principal-value sense (extraction of field point by a small disk) because integrand is nonsingular: $G(r)dS = e^{-jkr}drd\varphi/4\pi$. Limit from both sides gives the same field.

- $\oint(\nabla' \cdot \mathbf{J}_s)\nabla' G dS'$ can be understood in terms of surface charge: $\nabla' \cdot \mathbf{J}_s = -j\omega\varrho_s$. Field discontinuity in normal component: $\mathbf{n} \cdot (\mathbf{E}_+ - \mathbf{E}_-) = \varrho_s/\epsilon$. (For constant ϱ_s on plane $\mathbf{E}_\pm = \pm\mathbf{n}\varrho_s/2\epsilon$.) No discontinuity for tangential components: $\mathbf{E}_t = \mathbf{E}_{t+} = \mathbf{E}_{t-} \Rightarrow$ principal-value integral when $\mathbf{r} \in S$, because integral over small symmetric circular disk $\int \nabla' G dS' = \oint_c \mathbf{m}' G dc' = \int \mathbf{m}' d\varphi'/4\pi = 0$.

Field from surface sources 3

- $\oint \nabla' G \times \mathbf{M}_s dS'$ gives discontinuous tangential field: $\mathbf{E}_{t+} - \mathbf{E}_{t-} = \mathbf{n} \times \mathbf{M}_s$. (For a constant planar source $\mathbf{E}_\pm = \pm\mathbf{n} \times \mathbf{M}_s/2$.) When $\mathbf{r} \in S$, small symmetric disk integrates as in previous case to $= 0$ $\Rightarrow$ principal-value integral.

- Functions $\mathcal{E}[\mathbf{J}_s|\mathbf{M}_s]_S$, $\mathcal{H}[\mathbf{J}_s|\mathbf{M}_s]_S$ can be applied to find the electromagnetic fields for $\mathbf{r}$ outside S (no problem) and for $\mathbf{r} \in S$ by interpreting integrals in principal-value sense with symmetric disk. For asymmetric disk, extra terms must be added.

- Connections between values of $\mathcal{E}$, $\mathcal{H}$ for $\mathbf{r} \in S$ and of the fields $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ in the limit $\mathbf{r} \to S$ from the two sides are

$$\mathbf{E}(\mathbf{r}_{1,2}) = \mathcal{PV}\ \mathcal{E}[\mathbf{J}_s|\mathbf{M}_s]_S + \frac{\mathbf{n}_{1,2}}{2j\omega\epsilon}\nabla\cdot\mathbf{J}_s - \frac{\mathbf{n}_{1,2}}{2}\times\mathbf{M}_s,$$

$$\mathbf{H}(\mathbf{r}_{1,2}) = \mathcal{PV}\ \mathcal{H}[\mathbf{J}_s|\mathbf{M}_s]_S + \frac{\mathbf{n}_{1,2}}{2j\omega\mu}\nabla\cdot\mathbf{M}_s + \frac{\mathbf{n}_{1,2}}{2}\times\mathbf{J}_s,$$

Huygens' principle and scattering problem

- Huygens' sources $\mathbf{J}_{H1}$, $\mathbf{M}_{H1}$ give scattered fields in V_1 and cancel the incident fields $\mathbf{E}^i, \mathbf{H}^i$ in V_2. Points $\mathbf{r}_1, \mathbf{r}_2$ on each side of S:

$$\mathbf{E}^i + \mathcal{PV}\ \mathcal{E}[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S + \tfrac{\mathbf{n}_1}{2j\omega\epsilon}\nabla\cdot(\mathbf{n}_1 \times \mathbf{H}) + \tfrac{\mathbf{n}_1}{2}\times(\mathbf{n}_1 \times \mathbf{E}) = \mathbf{E}(\mathbf{r}_1),$$

$$\mathbf{H}^i + \mathcal{PV}\ \mathcal{H}[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S - \tfrac{\mathbf{n}_1}{2j\omega\mu}\nabla\cdot(\mathbf{n}_1 \times \mathbf{E}) + \tfrac{\mathbf{n}_1}{2}\times(\mathbf{n}_1 \times \mathbf{H}) = \mathbf{H}(\mathbf{r}_1),$$

$$\mathbf{E}^i + \mathcal{PV}\ \mathcal{E}[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S + \tfrac{\mathbf{n}_2}{2j\omega\epsilon}\nabla\cdot(\mathbf{n}_1 \times \mathbf{H}) + \tfrac{\mathbf{n}_2}{2}\times(\mathbf{n}_1 \times \mathbf{E}) = 0,$$

$$\mathbf{H}^i + \mathcal{PV}\ \mathcal{H}[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S - \tfrac{\mathbf{n}_2}{2j\omega\mu}\nabla\cdot(\mathbf{n}_1 \times \mathbf{E}) + \tfrac{\mathbf{n}_2}{2}\times(\mathbf{n}_1 \times \mathbf{H}) = 0.$$

- For $\mathbf{r} \in S$ (smooth surface) gives average of fields:

$$\mathbf{E}^i + \mathcal{PV}\ \mathcal{E}[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S = \frac{1}{2}\mathbf{E}(\mathbf{r}),$$

$$\mathbf{H}^i + \mathcal{PV}\ \mathcal{H}[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S = \frac{1}{2}\mathbf{H}(\mathbf{r}).$$

Integral equations for PEC scatterer

- PEC scatterer in V_2, boundary condition $\mathbf{n}\times\mathbf{E} = 0$ on S. Unknown $\mathbf{J}_s = \mathbf{n}_1 \times \mathbf{H}$

$$\mathbf{E}^i(\mathbf{r}) + \mathcal{PV}\ \mathcal{E}[\mathbf{n}_1 \times \mathbf{H}|0]_S = \frac{1}{2}\mathbf{E}(\mathbf{r}),$$

$$\mathbf{H}^i(\mathbf{r}) + \mathcal{PV}\ \mathcal{H}[\mathbf{n}_1 \times \mathbf{H}|0]_S = \frac{1}{2}\mathbf{H}(\mathbf{r}).$$

- Taking only tangential component $\Rightarrow$ integral equations on S:

$$-\mathbf{n}_1 \times \mathcal{PV}\ \mathcal{E}[\mathbf{J}_s|0]_S = \mathbf{n}_1 \times \mathbf{E}^i(\mathbf{r}), \qquad \text{EFIE},$$

$$\frac{1}{2}\mathbf{J}_s(\mathbf{r}) - \mathbf{n}_1 \times \mathcal{PV}\ \mathcal{H}[\mathbf{J}_s|0]_S = \mathbf{n}_1 \times \mathbf{H}^i(\mathbf{r}), \qquad \text{MFIE}.$$

- EFIE = electric field integral equation = Fredholm of 1st kind
- MFIE = magnetic field integral equation = Fredholm of 2nd kind

Integral equations for PEC scatterer 2

- More explicit forms for EFIE and MFIE (principal-value integrals)

$$j\omega\mu \mathbf{n}_1 \times \oint_S \left(G(\mathbf{r}-\mathbf{r}')\mathbf{J}_s(\mathbf{r}') + \frac{1}{k^2}\nabla' \cdot \mathbf{J}_s(\mathbf{r}')\nabla' G(\mathbf{r}-\mathbf{r}') \right) dS' = \mathbf{n}_1 \times \mathbf{E}^i(\mathbf{r}),$$

$$\frac{1}{2}\mathbf{J}_s(\mathbf{r}) - \mathbf{n}_1 \times \oint_S \mathbf{J}_s(\mathbf{r}') \times \nabla' G(\mathbf{r}-\mathbf{r}') dS' = \mathbf{n}_1 \times \mathbf{H}^i(\mathbf{r}).$$

- EFIE (Fredholm of 1st kind) contains differentiation of $\mathbf{J}_s \Rightarrow$ less efficient in general. MFIE (Fredholm of 2nd kind) numerically better behaved. Not suitable for thin objects because $\mathbf{J}_s$ and ∇G almost parallel, $\mathbf{J}_s \times \nabla G$ small quantity $\Rightarrow$ EFIE better.

- Problems of uniqueness: currents of resonator modes of any amplitude can be added if frequency equals resonator frequency. (Resonator = cavity of V_2). Linear combination of EFIE and MFIE may have complex resonances $\Rightarrow$ uniqueness for real ω.

Cavity resonator

- Taking V_2 resonator cavity, S = PEC boundary, Huygens' source formulations with no incident field are

$$j\omega\mu\mathbf{n}_2\times\mathcal{PV}\oint_S\left(G(\mathbf{r}-\mathbf{r}')\mathbf{J}_s(\mathbf{r}')+\frac{1}{k^2}\nabla'\cdot\mathbf{J}_s(\mathbf{r}')\nabla'G(\mathbf{r}-\mathbf{r}')\right)dS'=0,$$

$$\frac{1}{2}\mathbf{J}_s(\mathbf{r})-\mathbf{n}_2\times\mathcal{PV}\oint_S\mathbf{J}_s(\mathbf{r}')\times\nabla'G(\mathbf{r}-\mathbf{r}')dS'=0.$$

- Solutions $\mathbf{J}_s(\mathbf{r})\neq 0$ for certain frequencies (k values, G depends on k)

- Integral equation has similar form for interior and exterior problems!

- Resonances can be found from the scattering problem by considering singularities of the the moment matrix.

Scattered fields

- After solving for the Huygens' source $\mathbf{J}_s$, the scattered electric field in V_1 can be found through integration:

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \oint_S \overline{\overline{G}}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{J}_s(\mathbf{r}')dS'$$

- Resonator modes in $\mathbf{J}_s(\mathbf{r})$ do not affect the field $\mathbf{E}(\mathbf{r})$, because they are nonradiating sources

Dielectric scatterer

- Dielectric scatterer (homogeneous, ϵ_r constant) $V_2 \Rightarrow$ both $\mathbf{n}\times\mathbf{H}$ and $\mathbf{n}\times\mathbf{E}$ unknown on S
- Huygens' principle must be applied to V_1 and V_2 with respective Green functions (different media 1,2)
- For example, MFIE formulation for points $\mathbf{r}_1, \mathbf{r}_2 \to S$:

$$\frac{1}{2}\mathbf{n}_1 \times \mathbf{H}(\mathbf{r}_1) - \mathbf{n}_1 \times \mathcal{PV}\ \mathcal{H}_1[\mathbf{n}_1 \times \mathbf{H}| - \mathbf{n}_1 \times \mathbf{E}]_S = \mathbf{n}_1 \times \mathbf{H}^i,$$

$$\frac{1}{2}\mathbf{n}_2 \times \mathbf{H}(\mathbf{r}_2) - \mathbf{n}_2 \times \mathcal{PV}\ \mathcal{H}_2[\mathbf{n}_2 \times \mathbf{H}| - \mathbf{n}_2 \times \mathbf{E}]_S = 0,$$

- Pair of equations for two unknown functions.
- EFIE formulation in a corresponding way.

Problems

21 Express Huygens' principle in a transmission line satisfying the equations

$$\partial_z \begin{pmatrix} U(z) \\ I(z) \end{pmatrix} = -j \begin{pmatrix} 0 & \omega\ell \\ \omega c & 0 \end{pmatrix} \begin{pmatrix} U(z) \\ I(z) \end{pmatrix} + \begin{pmatrix} u(z) \\ i(z) \end{pmatrix},$$

where $u(z), i(z)$ are the voltage and current source functions. Assume all sources are in the region $z < 0$ and replace the line $z < 0$ by Huygens' sources at $z = 0$.

22 Extend the previous Huygens' principle by assuming a pulse function $P(z)$ as

$$P(z < 0) = 0, \quad P(z > z_o) = 1, \quad P(0 < z < z_o) = 1/2$$

and assuming sources are in the region $z < 0$.

S-96.510 Advanced Field Theory
12. Field and Source Decompositions

I.V.Lindell

Decomposition of field problems

- Decomposition = splitting a problem in two simpler problems
- Fields of decomposed problems have restricted polarizations (example: TE and TM decomposition: axial components missing)
- Decomposed fields do not 'see' some medium parameters and they can be replaced (example $\overline{\overline{\epsilon}} \cdot \mathbf{u}_z$ for a TE field with $E_z = 0$ can be replaced by any value)
- If media of a decomposed problem can be simplified, solution becomes easier
- Also some boundaries can be simplified in decomposition
- Drawback: simple source may decompose to more complicated sources (example: point source = sum of two line sources)

Simple operator factorization

- Factorization of operator leads to decomposition

$$(\partial_z^2 + k^2)f(z) = (\partial_z - jk)(\partial_z + jk)f(z) = g(z)$$

$$f(z) = \frac{1}{(\partial_z - jk)(\partial_z + jk)}g(z) = \frac{1}{2jk}\left(\frac{1}{\partial_z - jk} - \frac{1}{\partial_z + jk}\right)g(z)$$

- General solution is decomposed: $f(z) = f_+(z) + f_-(z)$

$$(\partial_z - jk)f_-(z) = (2jk)^{-1}g(z), \qquad (\partial_z + jk)f_+(z) = -(2jk)^{-1}g(z)$$

- Homogeneous boundary conditions must be added. Outgoing-wave condition at $z = \pm\infty$ requires $f_+(-\infty) = f_-(\infty) = 0$

- For $g(z) = \delta(z)$ solutions $f_+ \sim e^{-jkz}U(z)$ and $f_-(z) \sim U(-z)e^{jkz}$

TE/TM source decomposition in isotropic medium

- What are sources of TE and TM fields and how to decompose a given source?
- Helmholtz equations with scalar operators

$$(\nabla^2 + k^2)\mathbf{E}(\mathbf{r}) = j\omega\mu(\overline{\overline{I}} + (1/k^2)\nabla\nabla) \cdot \mathbf{J}(\mathbf{r}) + \nabla \times \mathbf{M}(\mathbf{r})$$

$$(\nabla^2 + k^2)\mathbf{H}(\mathbf{r}) = j\omega\epsilon(\overline{\overline{I}} + (1/k^2)\nabla\nabla) \cdot \mathbf{M}(\mathbf{r}) - \nabla \times \mathbf{J}(\mathbf{r})$$

- Take $\mathbf{u}_z\cdot$ component of equations. $E_z = 0$, $H_z = 0 \Rightarrow$ conditions for TE/TM sources

$$j\omega\mu(\mathbf{u}_z + (1/k^2)\partial_z\nabla) \cdot \mathbf{J}^{TE}(\mathbf{r}) - \nabla \cdot \mathbf{u}_z \times \mathbf{M}^{TE}(\mathbf{r}) = 0$$

$$j\omega\epsilon(\mathbf{u}_z + (1/k^2)\partial_z\nabla) \cdot \mathbf{M}^{TM}(\mathbf{r}) + \nabla \cdot \mathbf{u}_z \times \mathbf{J}^{TM}(\mathbf{r}) = 0$$

- Simplest examples of TE/TM sources:

$$\mathbf{J}^{TM}(\mathbf{r}) = \mathbf{u}_z J^{TM}(\mathbf{r}), \quad \mathbf{M}^{TE}(\mathbf{r}) = \mathbf{u}_z M^{TE}(\mathbf{r})$$

TE/TM decomposition of dipole 1

- How to decompose a dipole $\mathbf{J}(\mathbf{r}) = \mathbf{u}IL\delta(\mathbf{r})$?
- If $\mathbf{u} = \mathbf{u}_z \;\Rightarrow$ TM source, no TE field
- How to decompose a transverse dipole with $\mathbf{u}\cdot\mathbf{u}_z = 0$?
- Find $J^{TM}(\mathbf{r})$ such that $\mathbf{J}(\mathbf{r}) - \mathbf{u}_z J^{TM}(\mathbf{r})$ = TE source.
  Equation for $J^{TM}(\mathbf{r})$ from TE condition:

$$(k^2\mathbf{u}_z + \partial_z\nabla)\cdot[\mathbf{J}(\mathbf{r}) - \mathbf{u}_z J^{TM}(\mathbf{r})] = 0$$

$$\Rightarrow \quad (\partial_z^2 + k^2)J^{TM}(\mathbf{r}) = \partial_z\nabla\cdot\mathbf{J}(\mathbf{r}) = IL\delta'(z)\mathbf{u}\cdot\nabla\delta(\boldsymbol{\rho})$$

- Solution in terms of one-dimensional Green function

$$J^{TM}(\mathbf{r}) = -IL\partial_z\frac{e^{-jk|z|}}{2jk}\mathbf{u}\cdot\nabla\delta(\boldsymbol{\rho}) = \frac{IL}{2}\mathrm{sgn}(z)e^{-jk|z|}\mathbf{u}\cdot\nabla\delta(\boldsymbol{\rho})$$

TE/TM decomposition of dipole 2

- Decomposed transverse dipole: $\mathbf{J}(\mathbf{r}) = \mathbf{J}^{TE}(\mathbf{r}) + \mathbf{u}_z J^{TM}(\mathbf{r})$

$$J^{TM}(\mathbf{r}) = \frac{IL}{2}\mathrm{sgn}(z)e^{-jk|z|}\mathbf{u}\cdot\nabla\delta(\boldsymbol{\rho})$$

- TM-component = current wave on a two-wire transmission line parallel to z axis; source $= -I$, wire distance $= L$

- TE-component = original current minus TM-component = transmission-line with current wave reversed

Uniaxial Anisotropic Medium 1

- TE/TM field decomposition possible in uniaxial anisotropic media
  Plane-wave relations $\mathbf{E}\cdot\mathbf{B} = 0$ and $\mathbf{H}\cdot\mathbf{D} = 0$ lead to
  $$\begin{pmatrix} \epsilon_z & \epsilon_t \\ \mu_z & \mu_t \end{pmatrix} \begin{pmatrix} E_z H_z \\ \mathbf{E}_t \cdot \mathbf{H}_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
- $\epsilon_z\mu_t - \mu_z\epsilon_t = 0 \;\Rightarrow$ affine-isotropic medium
- $\epsilon_z\mu_t - \mu_z\epsilon_t \neq 0 \;\Rightarrow$, every plane wave satisfies $E_z H_z = 0$, $\mathbf{E}_t\cdot\mathbf{H}_t = 0$
- Plane waves either TE or TM to z independent of $\mathbf{k}$ vector
- Any linear combinations (integrals) of plane waves can be decomposed to TE and TM parts
- Any fields outside sources decomposable to TE and TM parts
- Problem: how to decompose given sources in uniaxially anisotropic media to radiate TE/TM fields?

Uniaxial Anisotropic Medium 2

- Helmholtz determinant equations (4th order) for axial fields

$$[\det\overline{\overline{H}}_e(\nabla)]E_z(\mathbf{r}) = \mathbf{u}_z \cdot \overline{\overline{H}}_e^{(2)T}(\nabla) \cdot [j\omega\mathbf{J}(\mathbf{r}) + \nabla \times \overline{\overline{\mu}}^{-1} \cdot \mathbf{M}(\mathbf{r})]$$

$$[\det\overline{\overline{H}}_m(\nabla)]H_z(\mathbf{r}) = \mathbf{u}_z \cdot \overline{\overline{H}}_m^{(2)T}(\nabla) \cdot [j\omega\mathbf{M}(\mathbf{r})\nabla \times \overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J}(\mathbf{r})]$$

- Operators on both sides have common operators $(k_t^2 = \omega^2\mu_t\epsilon_t)$

$$\det\overline{\overline{\mu}}\ \det\overline{\overline{H}}_e(\nabla) = \det\overline{\overline{\epsilon}}\ \det\overline{\overline{H}}_m(\nabla) = \omega^2 H_\epsilon(\nabla)H_\mu(\nabla)$$

$$\det\overline{\overline{\mu}}\ \mathbf{u}_z \cdot \overline{\overline{H}}_e^{(2)T}(\nabla) = H_\mu(\nabla)(\mathbf{u}_z \cdot \nabla\nabla + k_t^2\mathbf{u}_z)$$

$$\det\overline{\overline{\epsilon}}\ \mathbf{u}_z \cdot \overline{\overline{H}}_m^{(2)T}(\nabla) = H_\epsilon(\nabla)(\mathbf{u}_z \cdot \nabla\nabla + k_t^2\mathbf{u}_z)$$

$$H_\epsilon(\nabla) = \overline{\overline{\epsilon}} : \nabla\nabla + \epsilon_z k_t^2, \qquad H_\mu(\nabla) = \overline{\overline{\mu}} : \nabla\nabla + \mu_z k_t^2$$

- Canceling operators $H_\epsilon(\nabla)$, $H_\mu(\nabla)$, equations reduced to 2nd order

Uniaxial Anisotropic Medium 3

- Second-order equations for axial field components

$$H_\epsilon(\nabla)E_z(\mathbf{r}) = (\mathbf{u}_z \cdot \nabla\nabla + k_t^2\mathbf{u}_z) \cdot [j\omega\mathbf{J}(\mathbf{r}) + \nabla \times \overline{\overline{\mu}}^{-1} \cdot \mathbf{M}(\mathbf{r})]$$

$$H_\mu(\nabla)H_z(\mathbf{r}) = (\mathbf{u}_z \cdot \nabla\nabla + k_t^2\mathbf{u}_z) \cdot [j\omega\mathbf{M}(\mathbf{r})\nabla \times \overline{\overline{\epsilon}}^{-1} \cdot \mathbf{J}(\mathbf{r})]$$

- Vanishing right sides produces TE and TM fields (no source for E_z, H_z). Equations for TE/TM sources

$$j\omega\mu_t(\mathbf{u}_z \cdot \nabla\nabla + k_t^2\mathbf{u}_z) \cdot \mathbf{J}^{TE}(\mathbf{r}) + k_t^2\mathbf{u}_z \cdot \nabla \times \mathbf{M}^{TE}(\mathbf{r}) = 0$$

$$j\omega\epsilon_t(\mathbf{u}_z \cdot \nabla\nabla + k_t^2\mathbf{u}_z) \cdot \mathbf{M}^{TM}(\mathbf{r}) - k_t^2\mathbf{u}_z \cdot \nabla \times \mathbf{J}^{TM}(\mathbf{r}) = 0$$

- Same equations as for the isotropic medium with $\mu \to \mu_t, \epsilon \to \epsilon_t$
- Similar source decomposition as for the isotropic medium!

Uniaxial Anisotropic Medium 4

- Medium can be replaced by effective media for decomposed fields

$$\mathbf{D}^{TE} = \overline{\overline{\epsilon}} \cdot \mathbf{E}^{TE} = \epsilon_t \mathbf{E}^{TE} = (\epsilon_t \overline{\overline{I}}_t + \epsilon_z^{TE} \mathbf{u}_z \mathbf{u}_z) \cdot \mathbf{E}^{TE}$$

$$\mathbf{B}^{TE} = \overline{\overline{\mu}} \cdot \mathbf{H}^{TM} = \mu_t \mathbf{H}^{TM} = (\mu_t \overline{\overline{I}}_t + \mu_z^{TM} \mathbf{u}_z \mathbf{u}_z) \cdot \mathbf{H}^{TM}$$

- Parameters $\epsilon_z^{TE}, \mu_z^{TM}$ can be chosen so that the two effective media become affine isotropic:

$$\epsilon_z^{TE} = \mu_z \epsilon_t / \mu_t, \qquad \mu_z^{TM} = \epsilon_z \mu_t / \epsilon_t$$

- Effective medium dyadics $(\overline{\overline{\epsilon}}^{TE}, \overline{\overline{\mu}})$ and $(\overline{\overline{\epsilon}}, \overline{\overline{\mu}}^{TM})$ satisfy

$$\overline{\overline{\epsilon}}^{TE} = \epsilon_t \overline{\overline{\mu}} / \mu_t, \qquad \overline{\overline{\mu}}^{TM} = \mu_t \overline{\overline{\epsilon}} / \epsilon_t$$

- Green dyadics exist is affine-isotropic media in analytic form.

- Decomposed problems can be solved in effective affine-isotropic media more easily than in the original medium.

Uniaxial Bi-anisotropic Medium 1

- Bianisotropic with uniaxial $\overline{\overline{\epsilon}}, \overline{\overline{\mu}}$ and $\overline{\overline{\xi}} = \xi \mathbf{u}_z\mathbf{u}_z, \overline{\overline{\zeta}} = \zeta \mathbf{u}_z\mathbf{u}_z$
- Eliminating $\mathbf{E}_t \cdot \mathbf{H}_t$ from plane-wave orthogonality conditions

$$\mathbf{E}\cdot\mathbf{B} = 0,\ \mathbf{H}\cdot\mathbf{D} = 0 \quad \Rightarrow \quad \mu_t H_z D_z = \epsilon_t E_z B_z$$

- For convenience define an auxiliary parameter A

$$A = \frac{D_z}{\epsilon_t E_z} = \frac{B_z}{\mu_t H_z} = \frac{\epsilon_z E_z + \xi H_z}{\epsilon_t E_z} = \frac{\mu_z H_z + \zeta E_z}{\mu_t H_z}$$

- A can be solved by eliminating $E_z/H_z \Rightarrow$ two solutions $A_\pm$

$$(\epsilon_t A - \epsilon_z)(\mu_t A - \mu_z) = \xi\zeta$$

$$A_\pm = \frac{1}{2}\left(\frac{\mu_z}{\mu_t} + \frac{\epsilon_z}{\epsilon_t}\right) \pm \sqrt{\frac{1}{4}\left(\frac{\mu_z}{\mu_t} + \frac{\epsilon_z}{\epsilon_t}\right)^2 + \frac{\xi\zeta}{\mu_t\epsilon_t}}$$

Uniaxial Bi-anisotropic Medium 2

- The decomposed plane waves $\mathbf{E}_\pm, \mathbf{H}_\pm$ satisfy conditions

$$(\mathbf{u}_z \cdot \mathbf{E} - Z_+ \mathbf{u}_z \cdot \mathbf{H})(\mathbf{u}_z \cdot \mathbf{E} - Z_- \mathbf{u}_z \cdot \mathbf{H}) = 0$$

- They are defined by constant axial impedances

$$Z_\pm = \frac{E_\pm}{H_\pm} = \frac{A_\pm \mu_t - \mu_z}{\zeta} = \frac{\xi}{A_\pm \epsilon_t - \epsilon_z}$$

- Effective medium dyadics can be defined with $\overline{\overline{\xi}}_\pm = \overline{\overline{\zeta}}_\pm = 0$:

$$\mathbf{D}_\pm = [\epsilon_t \overline{\overline{I}}_t + (\epsilon_z \xi Z_\pm^{-1} \mathbf{u}_z \mathbf{u}_z] \cdot \mathbf{E}_\pm = \overline{\overline{\epsilon}}_\pm \cdot \mathbf{E}_\pm$$

$$\mathbf{B}_\pm = [\mu_t \overline{\overline{I}}_t + (\mu_z \zeta Z_\pm \mathbf{u}_z \mathbf{u}_z] \cdot \mathbf{H}_\pm = \overline{\overline{\mu}}_\pm \cdot \mathbf{E}_\pm$$

$$\overline{\overline{\epsilon}}_\pm = \epsilon_t \overline{\overline{A}}_\pm, \quad \overline{\overline{\mu}}_\pm = \mu_t \overline{\overline{A}}_\pm, \quad \overline{\overline{A}}_\pm = \overline{\overline{I}}_t + A_\pm \mathbf{u}_z \mathbf{u}_z$$

- Any field outside its source can be decomposed in $\pm$ fields with vanishing axial field functions $F_\pm(\mathbf{r}) = \mathbf{u}_z \cdot [\mathbf{E}_\pm(\mathbf{r}) - Z_\pm \mathbf{H}_\pm(\mathbf{r})] = 0$

Uniaxial Bi-anisotropic Medium 3

- After some algebra Helmholtz determinant can be factorized as

$$H(\nabla) = \det[\overline{\overline{H}}_m(\nabla)\cdot\overline{\overline{\mu}}^{-1}] = \det[\overline{\overline{H}}_e(\nabla)\cdot\overline{\overline{\epsilon}}^{-1}] = \frac{\omega^2}{\mu_z\mu_t\epsilon_z\epsilon_t}H_+(\nabla)H_-(\nabla)$$

$$H_\pm(\nabla) = \nabla_t^2 + A_\pm(\partial_z^2 + k_t^2)$$

- 4th order Helmholtz equations for the axial field functions $F_\pm(\mathbf{r})$

$$H(\nabla)F_\pm(\mathbf{r}) = \det[\overline{\overline{H}}_e(\nabla)\cdot\overline{\overline{\epsilon}}^{-1}]E_{z\pm}(\mathbf{r}) - Z_\pm\det[\overline{\overline{H}}_m(\nabla)\cdot\overline{\overline{\mu}}^{-1}]H_{z\pm}(\mathbf{r})$$

$$= \frac{\mathbf{u}_z}{\det\overline{\overline{\epsilon}}}\cdot\overline{\overline{H}}_e^{(2)T}(\nabla)\cdot[j\omega\mathbf{J} + (\nabla\times\overline{\overline{I}} - j\omega\xi\mathbf{u}_z\mathbf{u}_z)\cdot\mathbf{M}]$$

$$-Z_\pm\frac{\mathbf{u}_z}{\det\overline{\overline{\mu}}}\cdot\overline{\overline{H}}_m^{(2)T}(\nabla)\cdot[j\omega\mathbf{M} - (\nabla\times\overline{\overline{I}} - j\omega\zeta\mathbf{u}_z\mathbf{u}_z)\cdot\mathbf{J}]$$

Uniaxial Bi-anisotropic Medium 4

- 4th order equations for $F_\pm(\mathbf{r})$ can be reduced to 2nd order equations by eliminating operators from both sides (uniqueness of solutions assuming)

- $F_+(\mathbf{r}) = 0$ and $F_-(\mathbf{r}) = 0$ lead to conditions for sources $\mathbf{J}_\pm, \mathbf{M}_\pm$:

$$\mathbf{u}_z \cdot [(\nabla\nabla + k_t^2 \overline{\overline{I}}) \cdot (j\omega\mu_t \mathbf{J}_\pm - j\omega\epsilon_t Z_\pm \mathbf{M}_\pm) + k_t^2 \nabla \times (\mathbf{M}_\pm + Z_\pm \mathbf{J}_\pm)] = 0$$

- Example: axial sources $\mathbf{J}_\pm = \mathbf{u}_z J_\pm$, $\mathbf{M}_\pm = \mathbf{u}_z M_\pm$ satisfying

$$\mu_t J_\pm(\mathbf{r}) = E_t Z_\pm M_\pm(\mathbf{r})$$

- Given axial current $\mathbf{J}(\mathbf{r}) = \mathbf{u}_z J(\mathbf{r})$ can be decomposed as

$$\mathbf{J}_\pm(\mathbf{r}) = \pm \mathbf{u}_z \frac{Z_\pm}{Z_+ - Z_-} J(\mathbf{r}), \qquad \mathbf{M}_\pm(\mathbf{r}) = \pm \mathbf{u}_z \frac{\mu_t/\epsilon_t}{Z_+ - Z_-} J(\mathbf{r})$$

Bi-anisotropic Medium 1

- The previous theory can be generalized to a certain class of bi-anisotropic media called **decomposable media**
- Defining the medium: plane wave condition becomes

$$\mathbf{E}\cdot\mathbf{B}=0,\ \mathbf{H}\cdot\mathbf{D}=0 \quad \Rightarrow \quad (\mathbf{a}_1\cdot\mathbf{E}+\mathbf{a}_2\cdot\mathbf{H})(\mathbf{b}_1\cdot\mathbf{E}+\mathbf{b}_2\cdot\mathbf{H})=0$$

- Decomposition to a-waves and b-waves
- Decomposable media must be of the form

$$\overline{\overline{\epsilon}}=\frac{1}{2\tau}[-\overline{\overline{B}}^T+\mathbf{a}_2\mathbf{b}_1+\mathbf{b}_2\mathbf{a}_1],\quad \overline{\overline{\xi}}=\mathbf{x}\times\overline{\overline{I}}+\frac{1}{2\tau}[\mathbf{a}_2\mathbf{b}_2+\mathbf{b}_2\mathbf{a}_2],$$

$$\overline{\overline{\zeta}}=\mathbf{z}\times\overline{\overline{I}}+\frac{1}{2\eta}[\mathbf{a}_1\mathbf{b}_1+\mathbf{b}_1\mathbf{a}_1],\quad \overline{\overline{\mu}}=\frac{1}{2\eta}[\overline{\overline{B}}+\mathbf{a}_1\mathbf{b}_2+\mathbf{b}_1\mathbf{a}_2]$$

- See *IEEE Trans. Ant. Prop,* **46**(10)1584-1585,1998 for details.
- Can be generalized through duality transformation.

Bi-anisotropic Medium 2

- Decomposable media can be replaced by effective media for the a- and b-fields:

$$\overline{\overline{\epsilon}}_a = \frac{1}{2\tau}[-\overline{\overline{B}}^T + \mathbf{a}_2\mathbf{b}_1 - \mathbf{b}_2\mathbf{a}_1], \quad \overline{\overline{\xi}}_a = [\mathbf{x} - \frac{1}{2\tau}\mathbf{a}_2 \times \mathbf{b}_2] \times \overline{\overline{I}},$$

$$\overline{\overline{\zeta}}_a = [\mathbf{z} - \frac{1}{2\eta}\mathbf{a}_1 \times \mathbf{b}_1] \times \overline{\overline{I}}, \quad \overline{\overline{\mu}}_a = \frac{1}{2\eta}[\overline{\overline{B}} + \mathbf{a}_1\mathbf{b}_2 - \mathbf{b}_1\mathbf{a}_2]$$

$$\overline{\overline{\epsilon}}_b = \frac{1}{2\tau}[-\overline{\overline{B}}^T - \mathbf{a}_2\mathbf{b}_1 + \mathbf{b}_2\mathbf{a}_1], \quad \overline{\overline{\xi}}_b = [\mathbf{x} + \frac{1}{2\tau}\mathbf{a}_2 \times \mathbf{b}_2] \times \overline{\overline{I}},$$

$$\overline{\overline{\zeta}}_b = [\mathbf{z} + \frac{1}{2\eta}\mathbf{a}_1 \times \mathbf{b}_1] \times \overline{\overline{I}}, \quad \overline{\overline{\mu}}_b = \frac{1}{2\eta}[\overline{\overline{B}} - \mathbf{a}_1\mathbf{b}_2 + \mathbf{b}_1\mathbf{a}_2]$$

- The medium dyadics $\overline{\overline{\xi}}_a, \overline{\overline{\zeta}}_a, \overline{\overline{\xi}}_b, \overline{\overline{\zeta}}_b$ are antisymmetric and

$$\eta\overline{\overline{\mu}} + \tau\overline{\overline{\epsilon}}^T = \mathbf{a}_1\mathbf{b}_2 + \mathbf{b}_1\mathbf{a}_2, \quad \eta\overline{\overline{\mu}}_a + \tau\overline{\overline{\epsilon}}_a^T = \eta\overline{\overline{\mu}}_b + \tau\overline{\overline{\epsilon}}_b^T = 0$$

- Analytic Green dyadics can be constructed for the effective media!

Bi-anisotropic Medium 3

- Helmholtz determinants are factorizable in decomposable media!

$$H(\nabla) = \det[\overline{\overline{\mu}}^{-1}\cdot\overline{\overline{H}}_m(\nabla)] = \det[\overline{\overline{H}}_e(\nabla)\cdot\overline{\overline{\epsilon}}^{-1}] = -\frac{\omega^2\eta H_a(\nabla)H_b(\nabla)}{\tau\det\overline{\overline{\epsilon}}\ \det\overline{\overline{\mu}}}$$

$$H_a(\nabla) = \overline{\overline{\mu}}_a : (\nabla + j\omega\mathbf{z}_a)(\nabla - j\omega\mathbf{x}_a) - \omega^2(\eta/\tau)\det\overline{\overline{\mu}}_a$$
$$H_b(\nabla) = \overline{\overline{\mu}}_b : (\nabla + j\omega\mathbf{z}_b)(\nabla - j\omega\mathbf{x}_b) - \omega^2(\eta/\tau)\det\overline{\overline{\mu}}_b$$

- Source decomposition can be done in principle as for the uniaxial case, see *J. Electro Waves Appl.* **13**(4)429-444, 1999.

- Plane waves in decomposable media have two second-order wave surfaces (instead of one 4th order surface).

- Also Hertzian potentials can be generalized to decomposable media, see *J. Electro Waves Appl.* **15**(1)3-18,2001.

Problems

23 Check that the plane-wave fields in a decomposable medium defined by the medium equations on viewgraph 12.14 satisfy the conditions $(\mathbf{a}_1 \cdot \mathbf{E} + \mathbf{a}_2 \cdot \mathbf{H})(\mathbf{b}_1 \cdot \mathbf{E} + \mathbf{b}_2 \cdot \mathbf{H}) = 0$

24 Find the equations for the wave-vector $\mathbf{k}$ in the above medium using the effective medium concept. Show that the two wave-vector surfaces for each of the effective media coincide as one surface.

Vector formulas

General formulas

$$\nabla(\alpha f(\mathbf{r})) = \alpha \nabla f(\mathbf{r})$$
$$\nabla[f(\mathbf{r})g(\mathbf{r})] = g(\mathbf{r})\nabla f(\mathbf{r}) + f(\mathbf{r})\nabla g(\mathbf{r})$$
$$\nabla \cdot [\alpha \mathbf{f}(\mathbf{r})] = \alpha \nabla \cdot \mathbf{f}(\mathbf{r})$$
$$\nabla \cdot [f(\mathbf{r})\mathbf{g}(\mathbf{r})] = [\nabla f(\mathbf{r})] \cdot \mathbf{g}(\mathbf{r}) + f(\mathbf{r})[\nabla \cdot \mathbf{g}(\mathbf{r})]$$
$$\nabla \times [\alpha \mathbf{f}(\mathbf{r})] = \alpha \nabla \times \mathbf{f}(\mathbf{r})$$
$$\nabla \times [f(\mathbf{r})\mathbf{g}(\mathbf{r})] = [\nabla f(\mathbf{r})] \times \mathbf{g}(\mathbf{r}) + f(\mathbf{r})[\nabla \times \mathbf{g}(\mathbf{r})]$$
$$\nabla \cdot [\mathbf{f}(\mathbf{r}) \times \mathbf{g}(\mathbf{r})] = [\nabla \times \mathbf{f}(\mathbf{r})] \cdot \mathbf{g}(\mathbf{r}) - \mathbf{f}(\mathbf{r}) \cdot [\nabla \times \mathbf{g}(\mathbf{r})]$$
$$\nabla \times [\mathbf{f} \times \mathbf{g}] = \mathbf{f}[\nabla \cdot \mathbf{g}] - \mathbf{g}[\nabla \cdot \mathbf{f}] + [\mathbf{g} \cdot \nabla]\mathbf{f} - [\mathbf{f} \cdot \nabla]\mathbf{g}$$
$$\nabla \times \nabla f(\mathbf{r}) = 0$$
$$\nabla \cdot [\nabla \times \mathbf{f}(\mathbf{r})] = 0$$
$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - \nabla \cdot \nabla \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \nabla^2 \mathbf{f}$$

$$\int_V \nabla \cdot \mathbf{f} dV = \oint_S \mathbf{f} \cdot d\mathbf{S} \qquad \text{(Gauss)}$$

$$\int_S \nabla \times \mathbf{f} \cdot d\mathbf{S} = \oint_C \mathbf{f} \cdot d\ell \qquad \text{(Stokes)}$$

Cartesian coordinates $x,\ y,\ z$

$$\nabla f = \mathbf{u}_x \frac{\partial}{\partial x} f + \mathbf{u}_y \frac{\partial}{\partial y} f + \mathbf{u}_z \frac{\partial}{\partial z} f$$

$$\nabla \cdot \mathbf{f} = \frac{\partial}{\partial x} f_x + \frac{\partial}{\partial y} f_y + \frac{\partial}{\partial z} f_z$$

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical coordinates $\rho,\ \varphi,\ z$

$$\nabla f = \mathbf{u}_\rho \frac{\partial}{\partial \rho} f + \mathbf{u}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} f + \mathbf{u}_z \frac{\partial}{\partial z} f$$

$$\nabla \cdot \mathbf{f} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho f_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \varphi} f_\varphi + \frac{\partial}{\partial z} f_z$$

$$\nabla \times \mathbf{f} = \frac{1}{\rho}\begin{vmatrix} \mathbf{u}_\rho & \rho\mathbf{u}_\varphi & \mathbf{u}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ f_\rho & \rho f_\varphi & f_z \end{vmatrix}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla \boldsymbol{\rho} = \nabla(\mathbf{r} - \mathbf{u}_z z) = \overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z = \overline{\overline{I}}_t, \qquad \nabla \cdot \boldsymbol{\rho} = 2, \qquad \nabla \times \boldsymbol{\rho} = 0$$

$$\rho = |\boldsymbol{\rho}| = |\mathbf{r} - \mathbf{u}_z z|, \qquad \nabla \rho = \mathbf{u}_\rho$$

$$\nabla\nabla\rho = \nabla \mathbf{u}_\rho = \frac{1}{\rho}(\overline{\overline{I}} - \mathbf{u}_z\mathbf{u}_z - \mathbf{u}_\rho\mathbf{u}_\rho) = \frac{1}{\rho}\mathbf{u}_z\mathbf{u}_z {}^{\times}_{\times} \mathbf{u}_\rho\mathbf{u}_\rho = \frac{1}{\rho}\mathbf{u}_\varphi\mathbf{u}_\varphi$$

$$\nabla^2\rho = \nabla\cdot\mathbf{u}_\rho = \frac{1}{\rho}, \qquad \nabla\times\mathbf{u}_\rho = 0$$

$$\nabla\varphi = \frac{1}{\rho}\mathbf{u}_\varphi, \qquad \nabla\mathbf{u}_\varphi = -\frac{1}{\rho}\mathbf{u}_\varphi\mathbf{u}_\rho, \qquad \nabla\cdot\mathbf{u}_\varphi = 0, \qquad \nabla\times\mathbf{u}_\varphi = \frac{1}{\rho}\mathbf{u}_z$$

$$g_2(\boldsymbol{\rho}) = -\frac{1}{2\pi}\ln(k|\boldsymbol{\rho}|), \qquad \nabla g_2(\boldsymbol{\rho}) = -\frac{\mathbf{u}_\rho}{2\pi\rho}$$

$$\nabla\nabla g_2(\boldsymbol{\rho}) = \mathrm{PV}\frac{1}{2\pi\rho^2}(\mathbf{u}_\rho\mathbf{u}_\rho - \mathbf{u}_\varphi\mathbf{u}_\varphi) - \frac{1}{2}\overline{\overline{I}}_t\delta(\boldsymbol{\rho})$$

$$\nabla^2 g_2(\boldsymbol{\rho}) = -\delta(\boldsymbol{\rho})$$

$$\mathbf{u}_z\mathbf{u}_z{}^\times_\times\nabla\nabla g_2(\boldsymbol{\rho}) = -\mathrm{PV}\frac{1}{2\pi\rho^2}(\mathbf{u}_\rho\mathbf{u}_\rho - \mathbf{u}_\varphi\mathbf{u}_\varphi) - \frac{1}{2}\overline{\overline{I}}_t\delta(\boldsymbol{\rho})$$

$$G_2(\boldsymbol{\rho}) = \frac{1}{4j}H_o^{(2)}(k|\boldsymbol{\rho}|), \qquad \nabla G_2(\boldsymbol{\rho}) = -\mathbf{u}_\rho\frac{k}{4j}H_1^{(2)}(k|\boldsymbol{\rho}|)$$

$$\nabla\nabla G_2(\boldsymbol{\rho}) = -\mathbf{u}_\rho\mathbf{u}_\rho k^2 G_2(k|\boldsymbol{\rho}|) + \mathrm{PV}(\mathbf{u}_\rho\mathbf{u}_\rho - \mathbf{u}_\varphi\mathbf{u}_\varphi)\frac{k}{4j\rho}H_1^{(2)}(k|\boldsymbol{\rho}|) - \frac{1}{2}\overline{\overline{I}}_t\delta(\boldsymbol{\rho})$$

$$\nabla^2 G_2(\boldsymbol{\rho}) = \overline{\overline{I}} : \nabla\nabla G_2(\boldsymbol{\rho}) = -k^2 G_2(\boldsymbol{\rho}) - \delta(\boldsymbol{\rho})$$

Spherical coordinates $r,\ \theta,\ \varphi$

$$\nabla f = \mathbf{u}_r \frac{\partial}{\partial r} f + \mathbf{u}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} f + \mathbf{u}_\varphi \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} f$$

$$\nabla \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 f_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta f_\theta) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} f_\varphi$$

$$\nabla \times \mathbf{f} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{u}_r & r\mathbf{u}_\theta & r\sin\theta\mathbf{u}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ f_r & r f_\theta & r \sin\theta f_\varphi \end{vmatrix}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2}$$

$$\nabla \mathbf{r} = \overline{\overline{I}}, \qquad \nabla \cdot \mathbf{r} = 3, \qquad \nabla \times \mathbf{r} = 0$$

$$\nabla r = \mathbf{u}_r, \qquad \nabla \mathbf{u}_r = \frac{1}{r}(\overline{\overline{I}} - \mathbf{u}_r \mathbf{u}_r), \qquad \nabla \cdot \mathbf{u}_r = \frac{2}{r}, \qquad \nabla \times \mathbf{u}_r = 0$$

$$\nabla(\mathbf{a} \times \mathbf{r}) = -\overline{\overline{I}} \times \mathbf{a} = -\mathbf{a} \times \overline{\overline{I}}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{r}) = 0, \qquad \nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$$

$$G(\mathbf{r}) = \frac{e^{-jkr}}{4\pi r}, \qquad r = |\mathbf{r}|$$

$$\nabla G(\mathbf{r}) = -\frac{\mathbf{u}_r}{r}(1 + jkr)G(\mathbf{r})$$

$$\nabla\nabla G(\mathbf{r}) = -\mathbf{u}_r\mathbf{u}_r k^2 G(\mathbf{r}) - \mathrm{PV}\frac{1}{r^2}(1 + jkr)(\overline{\overline{I}} - 3\mathbf{u}_r\mathbf{u}_r)G(\mathbf{r}) - \frac{1}{3}\overline{\overline{I}}\delta(\mathbf{r})$$

$$\nabla^2 G(\mathbf{r}) = \overline{\overline{I}} : \nabla\nabla G(\mathbf{r}) = -k^2 G(\mathbf{r}) - \delta(\mathbf{r})$$

Dyadic identities

In the following table, $\overline{\overline{A}}, \overline{\overline{B}}, \overline{\overline{C}}, \overline{\overline{D}}$ denote arbitrary dyadics while $\overline{\overline{S}}$ is a symmetric dyadic. $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are arbitrary vectors and $\mathbf{u}$ is a unit vector.

Definitions

$$(\mathbf{ab}) \cdot (\mathbf{cd}) = (\mathbf{b} \cdot \mathbf{c})\mathbf{ad}$$

$$(\mathbf{ab}) : (\mathbf{cd}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$$

$$(\mathbf{ab}) {}^{\cdot}_{\times} (\mathbf{cd}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \times \mathbf{d})$$

$$(\mathbf{ab}) {}^{\times}_{\cdot} (\mathbf{cd}) = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$$

$$(\mathbf{ab}) {}^{\times}_{\times} (\mathbf{cd}) = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \times \mathbf{d})$$

$$\overline{\overline{I}} \cdot \mathbf{a} = \mathbf{a} \cdot \overline{\overline{I}} = \mathbf{a}$$

$$\overline{\overline{A}}^2 = \overline{\overline{A}} \cdot \overline{\overline{A}}, \qquad \overline{\overline{A}}^{-2} = (\overline{\overline{A}}^{-1})^2 = (\overline{\overline{A}}^2)^{-1}$$

$$\overline{\overline{A}}^{(2)} = \frac{1}{2}\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{A}}, \qquad \overline{\overline{A}}^{(-2)} = (\overline{\overline{A}}^{-1})^{(2)} = (\overline{\overline{A}}^{(2)})^{-1}$$

$$\mathrm{tr}\overline{\overline{A}} = \overline{\overline{A}} : \overline{\overline{I}}$$

$$\mathrm{spm}\overline{\overline{A}} = \mathrm{tr}\overline{\overline{A}}^{(2)} = \frac{1}{2}\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}} : \overline{\overline{I}}$$

$$\det\overline{\overline{A}} = \frac{1}{6}\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}} : \overline{\overline{A}}$$

$$\det\overline{\overline{A}} \neq 0 \quad \leftrightarrow \quad \overline{\overline{A}} \text{ complete}$$

$$\det\overline{\overline{A}} = 0 \quad \leftrightarrow \quad \overline{\overline{A}} \text{ planar}$$

$$\overline{\overline{A}}^{(2)} = 0 \quad \leftrightarrow \quad \overline{\overline{A}} \text{ linear.}$$

Identities

Basic identities

$$\overline{\overline{A}} : \overline{\overline{B}} = \overline{\overline{B}} : \overline{\overline{A}} = \overline{\overline{A}}^T : \overline{\overline{B}}^T = (\overline{\overline{A}} \cdot \overline{\overline{B}}^T) : \overline{\overline{I}} = (\overline{\overline{A}}^T \cdot \overline{\overline{B}}) : \overline{\overline{I}}$$

$$\mathbf{a} \times \overline{\overline{I}} = \overline{\overline{I}} \times \mathbf{a} \quad \text{antisymmetric dyadic}$$

$$(\mathbf{a} \times \overline{\overline{I}})^T = -\mathbf{a} \times \overline{\overline{I}}$$

$$\overline{\overline{S}}^T = \overline{\overline{S}}, \quad \overline{\overline{S}}{}^{\cdot}_{\times}\overline{\overline{I}} = 0,\ \overline{\overline{S}} \text{ symmetric dyadic}$$

$$(\mathbf{a} \times \overline{\overline{I}}) : \overline{\overline{S}} = 0,$$

$$(\mathbf{a} \times \overline{\overline{I}}) : (\mathbf{b} \times \overline{\overline{I}}) = (\mathbf{b} \times \overline{\overline{I}}) : (\mathbf{a} \times \overline{\overline{I}}) = 2\mathbf{a} \cdot \mathbf{b}$$

$$\overline{\overline{A}} : (\mathbf{a} \times \overline{\overline{B}}) = -(\mathbf{a} \times \overline{\overline{A}}) : \overline{\overline{B}} = (\mathbf{a} \times \overline{\overline{B}}) : \overline{\overline{A}}$$

$$\overline{\overline{A}} : (\overline{\overline{B}} \times \mathbf{a}) = -(\overline{\overline{A}} \times \mathbf{a}) : \overline{\overline{B}} = (\overline{\overline{B}} \times \mathbf{a}) : \overline{\overline{A}}$$

Double-cross products

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}} = \overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{A}} =$$

$$= \left[(\overline{\overline{A}} : \overline{\overline{I}})(\overline{\overline{B}} : \overline{\overline{I}}) - \overline{\overline{A}} : \overline{\overline{B}}^T\right] \overline{\overline{I}} - (\overline{\overline{A}} : \overline{\overline{I}})\overline{\overline{B}}^T - (\overline{\overline{B}} : \overline{\overline{I}})\overline{\overline{A}}^T + [\overline{\overline{A}} \cdot \overline{\overline{B}} + \overline{\overline{B}} \cdot \overline{\overline{A}}]^T$$

$$\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{I}} = (\overline{\overline{A}} : \overline{\overline{I}})\overline{\overline{I}} - \overline{\overline{A}}^T$$

$$\overline{\overline{A}}{}^{\times}_{\times}(\mathbf{a} \times \overline{\overline{I}}) = \mathbf{a}(\overline{\overline{A}}{}^{\cdot}_{\times}\overline{\overline{I}}) + \overline{\overline{I}} \times (\mathbf{a} \cdot \overline{\overline{A}})$$

$$\overline{\overline{I}}{}^{\times}_{\times}\overline{\overline{I}} = 2\overline{\overline{I}}$$

$$(\mathbf{a}\times\overline{\overline{I}})^{\times}_{\times}\overline{\overline{I}} = \mathbf{a}\times\overline{\overline{I}}$$

$$\overline{\overline{S}}^{\times}_{\times}\overline{\overline{I}} = -\overline{\overline{S}} \quad (\overline{\overline{S}}\ \text{symmetric, trace free})$$

$$(\mathbf{a}\times\overline{\overline{I}})^{\times}_{\times}(\mathbf{b}\times\overline{\overline{I}}) = \mathbf{ab}+\mathbf{ba}$$

$$\overline{\overline{A}}^{\times}_{\times}(\mathbf{a}\times\overline{\overline{I}}) = (\mathbf{a}\cdot\overline{\overline{A}})\times\overline{\overline{I}} - \mathbf{a}(\overline{\overline{I}}\,{}^{\cdot}_{\times}\overline{\overline{A}})$$

$$\overline{\overline{S}}^{\times}_{\times}(\mathbf{a}\times\overline{\overline{I}}) = (\overline{\overline{S}}\cdot\mathbf{a})\times\overline{\overline{I}} \quad (\overline{\overline{S}}\ \text{symmetric})$$

$$(\overline{\overline{A}}\times\mathbf{a})^{\times}_{\times}(\overline{\overline{B}}\times\mathbf{a}) = (\overline{\overline{A}}^{\times}_{\times}\overline{\overline{B}})\cdot\mathbf{aa}$$

$$(\mathbf{a}\times\overline{\overline{A}})^{\times}_{\times}(\mathbf{a}\times\overline{\overline{B}}) = \mathbf{aa}\cdot(\overline{\overline{A}}^{\times}_{\times}\overline{\overline{B}})$$

$$(\mathbf{a}\times\overline{\overline{I}})^{\times}_{\times}(\mathbf{a}\times\overline{\overline{I}}) = 2\mathbf{aa}$$

Multiple double-cross products

$$\overline{\overline{A}}^{\times}_{\times}(\overline{\overline{B}}^{\times}_{\times}\overline{\overline{C}}) = (\overline{\overline{A}}:\overline{\overline{C}})\overline{\overline{B}} + (\overline{\overline{A}}:\overline{\overline{B}})\overline{\overline{C}} - \overline{\overline{B}}\cdot\overline{\overline{A}}^T\cdot\overline{\overline{C}} - \overline{\overline{C}}\cdot\overline{\overline{A}}^T\cdot\overline{\overline{B}}$$

$$\overline{\overline{I}}^{\times}_{\times}(\overline{\overline{A}}^{\times}_{\times}\overline{\overline{B}}) = (\overline{\overline{A}}:\overline{\overline{I}})\overline{\overline{B}} + (\overline{\overline{B}}:\overline{\overline{I}})\overline{\overline{A}} - (\overline{\overline{A}}\cdot\overline{\overline{B}} + \overline{\overline{B}}\cdot\overline{\overline{A}})$$

$$\overline{\overline{I}}{}^{\times}_{\times}(\overline{\overline{I}}{}^{\times}_{\times}\overline{\overline{A}}) = \overline{\overline{A}} + (\overline{\overline{A}} : \overline{\overline{I}})\overline{\overline{I}}$$

$$\overline{\overline{I}}{}^{\times}_{\times}(\overline{\overline{I}}{}^{\times}_{\times}\overline{\overline{I}}) = 4\overline{\overline{I}}$$

$$(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}){}^{\times}_{\times}(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}) = 8(\overline{\overline{A}}{}^{(2)})^{(2)} = 8\mathrm{det}\overline{\overline{A}}\ \overline{\overline{A}}$$

$$(\overline{\overline{A}}{}^{(2)})^{(2)} = \overline{\overline{A}}\mathrm{det}\overline{\overline{A}}$$

$$\mathrm{det}(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}) = 8\mathrm{det}^2\overline{\overline{A}}$$

Mixed products

$$(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}}) \cdot (\overline{\overline{C}}{}^{\times}_{\times}\overline{\overline{D}}) = (\overline{\overline{A}} \cdot \overline{\overline{C}}){}^{\times}_{\times}(\overline{\overline{B}} \cdot \overline{\overline{D}}) + (\overline{\overline{A}} \cdot \overline{\overline{D}}){}^{\times}_{\times}(\overline{\overline{B}} \cdot \overline{\overline{C}})$$

$$(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}) \cdot (\overline{\overline{B}}{}^{\times}_{\times}\overline{\overline{B}}) = 2(\overline{\overline{A}} \cdot \overline{\overline{B}}){}^{\times}_{\times}(\overline{\overline{A}} \cdot \overline{\overline{B}})$$

$$(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{B}})^2 = (\overline{\overline{A}}{}^2){}^{\times}_{\times}(\overline{\overline{B}}{}^2) + (\overline{\overline{A}} \cdot \overline{\overline{B}}){}^{\times}_{\times}(\overline{\overline{A}} \cdot \overline{\overline{B}})$$

$$(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}})^2 = 2(\overline{\overline{A}}{}^2){}^{\times}_{\times}(\overline{\overline{A}}{}^2)$$

$$(\overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{I}})^2 = (\overline{\overline{A}}{}^2){}^{\times}_{\times}\overline{\overline{I}} + \overline{\overline{A}}{}^{\times}_{\times}\overline{\overline{A}}$$

Inverses

$$(\overline{\overline{A}} \,{}^\times_\times\, \overline{\overline{A}})^T \cdot \overline{\overline{A}} = \overline{\overline{A}} \cdot (\overline{\overline{A}} \,{}^\times_\times\, \overline{\overline{A}})^T = \frac{1}{3}(\overline{\overline{A}} \,{}^\times_\times\, \overline{\overline{A}} : \overline{\overline{A}})\overline{\overline{I}}$$

$$\overline{\overline{A}}^{(2)T} \cdot \overline{\overline{A}} = \overline{\overline{A}} \cdot \overline{\overline{A}}^{(2)T} = \det\overline{\overline{A}}\,\overline{\overline{I}}$$

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^{(2)T}}{\det\overline{\overline{A}}} = \frac{3(\overline{\overline{A}} \,{}^\times_\times\, \overline{\overline{A}})^T}{\overline{\overline{A}} \,{}^\times_\times\, \overline{\overline{A}} : \overline{\overline{A}}} \qquad (\overline{\overline{A}} \text{ complete})$$

$$\overline{\overline{A}}^{-1} = \frac{(\overline{\overline{A}} \,{}^\times_\times\, \overline{\overline{A}}^{(2)*})^T}{\overline{\overline{A}}^{(2)} : \overline{\overline{A}}^{(2)*}} \quad \text{(planar inverse)}$$

$$\overline{\overline{A}}^{-1} \cdot \overline{\overline{A}} = \overline{\overline{I}} - \frac{\overline{\overline{A}}^{(2)*} \cdot \overline{\overline{A}}^{(2)T}}{\overline{\overline{A}}^{(2)} : \overline{\overline{A}}^{(2)*}} \qquad (\overline{\overline{A}} \text{ planar})$$

$$\overline{\overline{A}}^{-1} = \frac{\overline{\overline{A}}^T \,{}^\times_\times\, \mathbf{uu}}{\mathrm{spm}\overline{\overline{A}}} \quad \text{(two − dimensional inverse)}$$

$$\overline{\overline{A}}^{-1} \cdot \overline{\overline{A}} = \overline{\overline{I}}_t = \overline{\overline{I}} - \mathbf{uu} \quad (\overline{\overline{A}} \text{ two − dimensional})$$

Invariants

$$(\overline{\overline{A}} {}^\times_\times \overline{\overline{B}}) : \overline{\overline{C}} = \overline{\overline{A}} : (\overline{\overline{B}} {}^\times_\times \overline{\overline{C}}) \quad \text{(with all permutations)}$$

$$\text{spm}\overline{\overline{A}} = \text{tr}\overline{\overline{A}}^{(2)} = \frac{1}{2}[(\overline{\overline{A}} : \overline{\overline{I}})^2 - \overline{\overline{A}} : \overline{\overline{A}}^T]$$

$$\text{spm}(\overline{\overline{A}} \cdot \overline{\overline{B}}) = \text{tr}(\overline{\overline{A}}^{(2)} \cdot \overline{\overline{B}}^{(2)}) = \overline{\overline{A}}^{(2)} : \overline{\overline{B}}^{(2)T}$$

$$\det\overline{\overline{A}} = \frac{1}{3}\overline{\overline{A}}^3 : \overline{\overline{I}} - \frac{1}{2}(\overline{\overline{A}}^2 : I)(\overline{\overline{A}} : \overline{\overline{I}}) + \frac{1}{6}(\overline{\overline{A}} : \overline{\overline{I}})^3$$

$$\det(\overline{\overline{A}} {}^\times_\times \overline{\overline{A}}) = 8\det(\overline{\overline{A}}^{(2)}) = 8(\det\overline{\overline{A}})^2$$

$$\det(\overline{\overline{A}} \cdot \overline{\overline{B}}) = \det\overline{\overline{A}} \, \det\overline{\overline{B}}$$

$$\det(\overline{\overline{A}} \cdot \overline{\overline{B}} + \alpha\overline{\overline{I}}) = \det(\overline{\overline{B}} \cdot \overline{\overline{A}} + \alpha\overline{\overline{I}})$$

Other identities

$$\mathbf{a} \times (\overline{\overline{A}} {}^\times_\times \overline{\overline{B}}) = \overline{\overline{B}} \times (\mathbf{a} \cdot \overline{\overline{A}}) + \overline{\overline{A}} \times (\mathbf{a} \cdot \overline{\overline{B}})$$

$$(\overline{\overline{A}} {}^\times_\times \overline{\overline{B}}) \times \mathbf{a} = (\overline{\overline{A}} \cdot \mathbf{a}) \times \overline{\overline{B}} + (\overline{\overline{B}} \cdot \mathbf{a}) \times \overline{\overline{A}}$$

$$(\overline{\overline{A}} \cdot \mathbf{a}) \times (\overline{\overline{A}} \cdot \mathbf{b}) = \frac{1}{2}(\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{A}}) \cdot (\mathbf{a} \times \mathbf{b}) = \overline{\overline{A}}^{(2)} \cdot (\mathbf{a} \times \mathbf{b})$$

$$(\mathbf{a} \cdot \overline{\overline{A}}) \times (\mathbf{b} \cdot \overline{\overline{A}}) = \frac{1}{2}(\mathbf{a} \times \mathbf{b}) \cdot (\overline{\overline{A}} {}^{\times}_{\times} \overline{\overline{A}}) = (\mathbf{a} \times \mathbf{b}) \cdot \overline{\overline{A}}^{(2)}$$

$$\mathbf{a} \times \overline{\overline{A}}^{-1} = \frac{1}{\det \overline{\overline{A}}} \overline{\overline{A}}^{T} \times (\overline{\overline{A}} \cdot \mathbf{a})$$

$$\overline{\overline{A}}^{-1} \times \mathbf{a} = \frac{1}{\det \overline{\overline{A}}} (\mathbf{a} \cdot \overline{\overline{A}}) \times \overline{\overline{A}}^{T}$$